(A)

Commutative Algebra Lecture 29 Tues 12/5/09

Rings of fractions

$$\mathbb{Q} = \{ x/y \mid x, y \in \mathbb{Z},\ y \neq 0 \}$$

$$F(x) = \{ p(x)/q(x) \mid p(x), q(x) \in F[x],\ q(x) \neq 0 \}$$

note round brackets (field)

restriction on denominators

really equivalence classes

$$\tfrac{1}{2} = \tfrac{3}{6} = \tfrac{5}{10} = \cdots$$

Operations on fractions:

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw}$$

$$\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$$

denominators closed under multiplication

Also want

$$z \equiv z/1$$

1 as a denominator

(B)

In general, let A be a ring and $S \subseteq A$ multiplicatively closed, meaning

$$\boxed{1 \in S \text{ and } (\forall x, y \in S)\ xy \in S},$$

Examples: (1) $P \underset{\text{prime}}{\triangleleft} A$, $S = A \setminus P$

(2) special case of (1), A an integral domain and $S = A \setminus \{0\}$.

(3) $x \in A$, $S = \{1, x, x^2, x^3, \dots\}$

(4) S = group of units of A

(5) $A = \mathbb{Z}_6$,

$S_1 = \{1, 5\}$ = group of units

$S_2 = \{1, 2, 4\}$ (powers of 2)

$S_3 = \{1, 3\}$ (" " 3),

$S_4 = \{1, 3, 5\} = \mathbb{Z}_6 \setminus 2\mathbb{Z}_6$

$S_5 = \{1, 2, 4, 5\} = \mathbb{Z}_6 \setminus 3\mathbb{Z}_6$

complements of prime ideals

(c)

Form $A \times S = \{(a,s) \mid a \in A, s \in S\}$

numerators (everything)

denominators (restricted)

and an appropriate equivalence relation?

In $\mathbb{Q}$, $\frac{a}{b} = \frac{c}{d} \iff ad - bc = 0$

suggesting $\boxed{(a,b) \equiv (c,d) \iff ad - bc = 0}$?

? transitivity: $\underbrace{(a,s) \equiv (b,t)}_{at - bs = 0} \underbrace{\equiv (c,u)}_{bu - ct = 0}$

$\Rightarrow atu - bsu = 0 = bus - cts$

$\Rightarrow atu = bsu = cts$

$\Rightarrow (au - cs)\,t = 0$

may not be able to cancel in A

Define
$$(a,s) \equiv (b,t)$$
iff $\boxed{(at-bs)u = 0 \quad \exists u \in S}$

Then $\equiv$ is reflexive, symmetric & transitive

Define
$$a/s \overset{def}{=} \text{equivalence class of } (a,s)$$

and put
$$S^{-1}A = \{ a/s \mid a \in A,\ s \in S \}$$

with operations
$$a/s + b/t = \frac{at+bs}{st}$$
$$a/s \; b/t = ab/st,$$

both well-defined (some effort to check).

Routine to check: $S^{-1}A$ is a ring with zero $0/s$ and identity $1 = s/s$ $\forall s \in S$

called the <u>ring of fractions of A with respect to S</u>.

Let $f : A \to S^{-1}A$, $a \mapsto a/1$,

easily seen to be a ring hom

$\boxed{f \text{ is injective} \iff S \text{ contains no zero divisors}}$

$a \in \ker f \iff a/1 = 0/1$

$\iff (a1 - 0\,1)u = au = 0 \quad \exists u \in S$

Examples: (1) $f : \mathbb{Z} \to \mathbb{Q}$
$f : F[x] \to F(x)$ } embeddings

(2) $A = \mathbb{Z}_6$, $S_1 = \{1, 5\}$, $S_2 = \{1, 2, 4\}$, $S_3 = \{1, 3\}$.

$$S_1^{-1}A = \{0, 1, 2, 3, 4, 5, \; 0/5, 1/5, 2/5, 3/5, 4/5, 5/5\}$$

$0/5 = 0$, $1/5 = 5$, $2/5 = 4$, $3/5 = 3$, $4/5 = 2$, $5/5 = 1$

$\cong A$

$\boxed{\text{In general } G^{-1}A \cong A \text{ where } G = \text{group of units of } A.}$

$\boxed{F}$

$$S_3^{-1}\mathbb{Z}_6 = \{ \overset{0}{\overset{\|}{\tfrac{0}{1}}}, \overset{1}{\overset{\|}{\tfrac{1}{1}}}, \overset{0}{\overset{\|}{\tfrac{2}{1}}}, \tfrac{3}{1}, \tfrac{4}{1}, \tfrac{5}{1},$$

$$\tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{4}{3}, \tfrac{5}{3} \} = \{0, 1\}$$

$$\tfrac{2}{1} = \tfrac{0}{1} \text{ since } (2-0)\underset{\text{in } S_3}{\underset{\uparrow}{3}} = 0$$

$f : \mathbb{Z}_6 \to S_3^{-1}\mathbb{Z}_6$ has kernel $2\mathbb{Z}_6$

and f is onto, so

$$S_3^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_6 / 2\mathbb{Z}_6 \cong \mathbb{Z}_2 .$$

Similarly

$$S_2^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_6 / 3\mathbb{Z}_6 \cong \mathbb{Z}_3 .$$

(3) $\underset{\text{prime}}{P \triangleleft A}$, $S = A \setminus P$.

Form
$$A_P = S^{-1}A, \quad \underline{\text{localisation}} \text{ at } P.$$

Then A_P is a local ring with unique maximal ideal

$$M = \{ a/s \mid a \in P \}$$

proof easy using Part I technique

e.g. $A = \mathbb{Z}$, $P = 2\mathbb{Z}$,

$$A_P = \{ a/b \mid a, b \in \mathbb{Z}, \ b \text{ odd} \}.$$

Universal property (capturing idea of "division" by elements of S):

If $g : A \to B$ is a ring hom such that $g(s)$ is a unit of B for all $s \in S$ then $\exists !$ hom h such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & S^{-1}A \arrow[d, "h"] \\ & B \end{tikzcd}$$

commutes

$$\boxed{h : a/s \longmapsto g(a)\, g(s)^{-1}}$$

Up to isomorphism, $B \cong S^{-1}A$ is characterised by 3 properties of $g : A \to B$ (and then h becomes an isomorphism):

(1) $(\forall s \in S)$ $g(s)$ is a unit in B

(2) $\ker g = \{ a \in A \mid as = 0 \ \exists s \in S \}$

(3) $(\forall b \in B)$ $b = g(a)\, g(s)^{-1}$ $(\exists a \in A)(\exists s \in S)$

injectivity of h

surjectivity of h

(H)

Check these hold for $g = f : A \to S^{-1}A$, $a \mapsto a/1$:

✓ (1) $f(s) = s/1 = (1/s)^{-1}$ in $S^{-1}A$ $\forall s \in S$

✓ (2) $\ker f = \{a \in A \mid a/1 = 0/1\}$

$= \{a \in A \mid as = 0 \ \exists s \in S\}$

✓ (3) $(\forall\, a/s \in S^{-1}A)$ $a/s = a/1 \cdot 1/s = a/1\,(s/1)^{-1}$.

Examples: (i) $S^{-1}\mathbb{Z} \cong \mathbb{Q}$ where $S = \mathbb{Z} \setminus \{0\}$

under inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

(ii) $S^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_3$ where $S = \{1, 2\}$

under $g : \mathbb{Z}_6 \to \mathbb{Z}_3$ mod 3 hom

$0, 3 \mapsto 0$
$1, 4 \mapsto 1$
$2, 5 \mapsto 2$

✓ (1) $g(1), g(2)$ units of $\mathbb{Z}_3$

✓ (2) $\ker g = \{0, 3\} = \{a \in \mathbb{Z}_6 \mid 2a = 0\}$

✓ (3) holds trivially since g onto