Commutative Algebra   Lecture 27   Wed 6/5/09

Exactness properties of Tensor Product

Theorem: M, N, P are A-modules.

$$\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P))$$

as A-modules

$$\mathrm{Hom}(M\otimes N, P) \xrightarrow{\Phi} \mathrm{Hom}(M, \mathrm{Hom}(N,P)) \checkmark$$

$$f \longmapsto \Phi(f) \checkmark$$

$M \otimes N \to P$ $\qquad$ $M \longrightarrow \mathrm{Hom}(N, P)$

$x \otimes y \mapsto f(x\otimes y)$ $\qquad$ $x \longmapsto [\Phi(f)](x) \checkmark$

$N \longrightarrow P$

$y \longmapsto \underbrace{\{[\Phi(f)](x)\}(y)}_{= f(x\otimes y)}$

Φ one-one : routine ✓

onto : easy but goes via bilinear map $: M \times N \to P$

<u>Example</u>: $F[x,y] \cong F[x] \otimes_F F[x]$, so

$$\mathrm{Hom}(F[x,y], F) \cong \mathrm{Hom}(F[x] \otimes_F F[x], F)$$

$$\cong \mathrm{Hom}(F[x], \mathrm{Hom}(F[x], F))$$

e.g. Evaluating polys at $x = \alpha$, $y = \beta$ yields a linear transf: $F[x,y] \xrightarrow{e} F$

$$p(x,y) \longmapsto p(\alpha, \beta)$$

Φ breaks the evaluation into 2 steps

essentially: $p(x,y) \longmapsto p(\alpha, y) \longmapsto p(\alpha, \beta)$

$e(x^i \otimes x^j) = \alpha^i \beta^j$ effect on generators

$$\Phi(e) : x^i \longmapsto \Phi(e)(x^i)$$

$$x^j \longmapsto \left(\Phi(e)(x^i)\right)(x^j)$$

$$= e(x^i \otimes x^j)$$

$$= \alpha^i \beta^j$$

generators

(c)

<u>Example</u>: What is

$$\mathrm{Hom}(\mathbb{Z}_{15}, \mathbb{Z}_6) \; ?$$

$$\mathrm{Hom}(\mathbb{Z}_{15}, \mathbb{Z}_6) \cong \mathrm{Hom}(\mathbb{Z}_{15}, \mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_6))$$

$$\cong \mathrm{Hom}(\mathbb{Z}_{15} \otimes \mathbb{Z}_6, \mathbb{Z}_6)$$

$$\cong \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_3)$$

$$\cong \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_2) \oplus \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_3)$$

$$\cong 0 \oplus \mathbb{Z}_3 = \mathbb{Z}_3 .$$

More generally

$$\boxed{\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d \quad \text{where } d = \text{g.c.d}(m, n)}$$

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Theorem: If $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact, then $\forall$ A-modules N

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0$$

is exact.

$$M' \xrightarrow{f} M \qquad M' \otimes N \xrightarrow{f \otimes 1} M \otimes N$$

$$x \otimes y \longmapsto f(x) \otimes y$$

Examples:

(1) $2\mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0$

becomes, using $\mathbb{Z}_2$,

$$\underbrace{2\mathbb{Z} \otimes \mathbb{Z}_2}_{0} \hookrightarrow \underbrace{\mathbb{Z} \otimes \mathbb{Z}_2}_{\mathbb{Z}_2} \twoheadrightarrow \underbrace{(\mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}_2}_{\mathbb{Z}_2} \to 0$$

$$0 \hookrightarrow \mathbb{Z}_2 \twoheadrightarrow \mathbb{Z}_2 \to 0$$

$2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0$

$\textcircled{I}$ evaluation at 0

$$p(x) \longmapsto p(0)$$

(2)

$$xF[x] \hookrightarrow F[x] \twoheadrightarrow F \to 0$$

becomes, using $F[x]$,

$$\underbrace{xF[x] \otimes_F F[x]}_{xF[x,y]} \hookrightarrow \underbrace{F[x] \otimes_F F[x]}_{F[x,y]} \twoheadrightarrow \underbrace{F \otimes_F F[x]}_{F[y]} \to 0$$

$$p(x,y) \longmapsto p(0,y)$$

extension splits

$$F[x,y] = xF[x,y] + F[y]$$
$$\cong xF[x,y] \oplus F[y]$$

(3) $$\mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

becomes, using $\mathbb{Q}/\mathbb{Z}$,

$$\underbrace{\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}}_{\mathbb{Q}/\mathbb{Z}} \xrightarrow{0} \underbrace{\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}}_{0} \twoheadrightarrow \underbrace{\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}}_{0} \to 0$$

not injective

$\mathbb{Q}/\mathbb{Z}$ is nonflat

$$q \otimes \frac{x}{y} + \mathbb{Z}$$
$$= \frac{yq}{y} \otimes \frac{x}{y} + \mathbb{Z}$$
$$= \frac{q}{y} \otimes x + \mathbb{Z} = 0$$
$$= q/y \otimes \mathbb{Z}$$

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Proof of Theorem:

exact

$$\begin{tikzcd}
M' \arrow[d, "f"'] \\
M \arrow[d, "g"'] \\
M'' \arrow[d] \\
0
\end{tikzcd}$$

Assignment question

$\Longrightarrow$ exact $(\forall N, P)$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}(M'', \mathrm{Hom}(N,P)) \arrow[d] \arrow[r, phantom, "\cong"] & \mathrm{Hom}(M'' \otimes N, P) \arrow[d] \\
\mathrm{Hom}(M, \mathrm{Hom}(N,P)) \arrow[d] \arrow[r, phantom, "\cong"] & \mathrm{Hom}(M \otimes N, P) \arrow[d] \\
\mathrm{Hom}(M', \mathrm{Hom}(N,P)) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(M' \otimes N, P)
\end{tikzcd}$$

$\Rightarrow$ exact ($\forall N$)

$$\begin{array}{c} M'\otimes N \\ \downarrow \\ M\otimes N \\ \downarrow \\ M''\otimes N \end{array}$$

$\square$

Assignment question in reverse