

Exactness properties of Tensor Product

Theorem: M, N, P are A -modules.

$$\text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \text{Hom}(N, P))$$

as A -modules

$$\text{Hom}(M \otimes N, P) \xrightarrow{\Phi} \text{Hom}(M, \text{Hom}(N, P))$$

$$f \mapsto \Phi(f)$$

$$M \otimes N \rightarrow P$$

$$M \rightarrow \text{Hom}(N, P)$$

$$x \otimes y \mapsto f(x \otimes y)$$

$$x \mapsto [\Phi(f)](x)$$

$$N \rightarrow P$$

$$y \mapsto \{[\Phi(f)](x)\}(y)$$

$$f(x \otimes y)$$

Φ one-one : routine ✓

onto : easy but goes via bilinear map
 $f: M \times N \rightarrow P$

(B)

Example: $F[x, y] \cong F[x] \otimes_F F[y]$, so

$$\begin{aligned} \text{Hom}(F[x, y], F) &\cong \text{Hom}(F[x] \otimes_F F[y], F) \\ &\cong \text{Hom}(F[x], \text{Hom}(F[y], F)) \end{aligned}$$

eg. Evaluating polys at $x=\alpha, y=\beta$ yields a

linear transf: $F[x, y] \xrightarrow{e} F$

$$p(x, y) \mapsto p(\alpha, \beta)$$

Φ breaks the evaluation into 2 steps

essentially:

$$p(x, y) \mapsto p(\alpha, y) \mapsto p(\alpha, \beta)$$

$$e(x^i \otimes x^j) = \alpha^i \beta^j$$

effect on
generators

$$\Phi(e): x^i \mapsto \Phi(e)(x^i)$$

$$x^i \mapsto (\Phi(e)(x^i)) (x^j)$$

$$= e(x^i \otimes x^j)$$

$$= \alpha^i \beta^j$$

generators

(c)

Example: What is

$$\text{Hom}(\mathbb{Z}_{15}, \mathbb{Z}_6) \quad ?$$

$$\text{Hom}(\mathbb{Z}_{15}, \mathbb{Z}_6) \cong \text{Hom}(\mathbb{Z}_{15}, \text{Hom}(\mathbb{Z}_6, \mathbb{Z}_6))$$

$$\cong \text{Hom}(\mathbb{Z}_{15} \oplus \mathbb{Z}_6, \mathbb{Z}_6)$$

$$\cong \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_3)$$

$$\cong \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_2) \oplus \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_3)$$

$$\cong 0 \oplus \mathbb{Z}_3 = \mathbb{Z}_3.$$

More generally

$$\left(\begin{array}{l} \text{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d \\ \text{where } d = \text{g.c.d.}(m, n) \end{array} \right)$$

(D)

Theorem: If $M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$

is exact, then $\forall$ A -modules N

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \rightarrow 0$$

is exact.

$$M' \xrightarrow{f} M$$

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N$$

$$x \otimes y \mapsto f(x) \otimes y$$

Examples:

$$(1) \quad \mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

becomes, using $\mathbb{Z}_2$,

$$\mathbb{Z} \otimes \mathbb{Z}_2 \hookrightarrow \mathbb{Z} \otimes \mathbb{Z}_2 \rightarrow (\mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}_2 \rightarrow 0$$

$$0 \hookrightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 0$$

$$\mathbb{Z} \otimes 1 = 1 \otimes \mathbb{Z} = 1 \otimes 0 = 0$$

(I)

evolution at 0

$$p(x) \mapsto p(0)$$

$$(2) \quad x F[x] \hookrightarrow F[x] \twoheadrightarrow F \rightarrow 0$$

becomes, using $F[x]$,

$$x F[x] \otimes_F F[x] \hookrightarrow F[x] \otimes_F F[x] \twoheadrightarrow F \otimes_F F[x] \rightarrow 0$$

$$\underbrace{x F[x-y]} \hookrightarrow \underbrace{F[x-y]} \twoheadrightarrow \underbrace{F[y]} \rightarrow 0$$

$$p(x,y) \mapsto p(0,y)$$

extension split

$$F[x,y] = x F[x,y] + F[y]$$

$$\cong x F[x,y] \oplus F[y]$$

$$(3) \quad \mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

becomes, using $\mathbb{Q}/\mathbb{Z}$,

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

$$\underbrace{\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}}_{\mathbb{Q}/\mathbb{Z}} \xrightarrow{0} \underbrace{\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}}_0 \xrightarrow{0} \underbrace{\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}}_0 \rightarrow 0$$

not injective

$\mathbb{Q}/\mathbb{Z}$ is nonflat

$$q \otimes \frac{x}{y} + \mathbb{Z}$$

$$= yq \otimes \frac{x}{y} + \mathbb{Z}$$

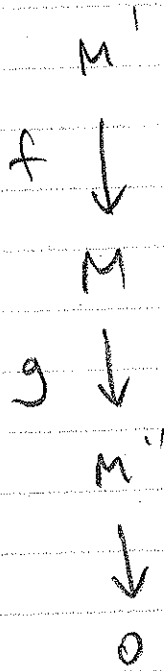
$$= \frac{q}{y} \otimes x + \mathbb{Z}$$

$$= q/y \otimes \mathbb{Z} = 0$$

F

Proof of Theorem :

exact



Assignment question

$\Rightarrow$ exact

$(\forall N, P)$

$$\text{Hom}(M'', \text{Hom}(N, P)) \cong \text{Hom}(M'' \otimes N, P)$$

$$\text{Hom}(M, \text{Hom}(N, P)) \cong \text{Hom}(M \otimes N, P)$$

$$\text{Hom}(M', \text{Hom}(N, P)) \cong \text{Hom}(M' \otimes N, P)$$

4

$\Rightarrow$

exit
(VN)

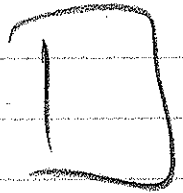
M'EN

↓

MEN

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M"EN



Assignment
question is
reverse