(A)

Commutative Algebra — Lecture 28 — Mon 12/5/09

Correction to example from last time:

$$0 \longrightarrow 2\mathbb{Z} \xhookrightarrow{i} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

becomes

$$\begin{array}{ccccccc} 2\mathbb{Z}\otimes\mathbb{Z}_2 & \xrightarrow{i\otimes 1} & \mathbb{Z}\otimes\mathbb{Z}_2 & \xrightarrow{q\otimes 1} & \mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}_2 & \longrightarrow & 0 \\ \cong & & \cong & & \cong & & \\ \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

$$i\otimes 1 : \underbrace{2\otimes 1}_{\substack{\text{nonzero}\\ \text{generator}\\ \text{in } 2\mathbb{Z}\otimes\mathbb{Z}_2}} \longmapsto \underbrace{2\otimes 1 = 1\otimes 2 = 1\otimes 0 = 0}_{\text{in } \mathbb{Z}\otimes\mathbb{Z}_2}$$

injectivity not preserved

$\mathbb{Z}_2$ is another non flat module

$\boxed{\text{flat modules preserve all of the short exactness}}$

Toy "template" proof:

$$\boxed{\text{Claim: } n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2 \quad \forall n \in \mathbb{Z}^+}$$

Proof:

$$\begin{tikzcd}
n\mathbb{Z}\times\mathbb{Z}_2 \arrow[r] \arrow[dr, "F"'] & n\mathbb{Z}\otimes\mathbb{Z}_2 \arrow[d, shift left, "f"] \\
& \mathbb{Z}_2 \arrow[u, bend left, "h"]
\end{tikzcd}$$

$$(nk, x) \overset{\text{bilinear}}{\longmapsto} kx \qquad nk\otimes x \longmapsto kx$$

Let $h : \mathbb{Z}_2 \longrightarrow n\mathbb{Z}\otimes\mathbb{Z}_2$

$$x \longmapsto n\otimes x$$

Then $f\circ h\,(x) = f(n\otimes x) = x$ $\quad f\circ h = 1_{\mathbb{Z}_2}$

$$h\circ f\,(nk\otimes x) = h(kx)$$
$$= n\otimes kx$$
$$= k(n\otimes x) = nk\otimes x$$

so $h\circ f$ fixes generators, so $h\circ f = 1_{n\mathbb{Z}\otimes\mathbb{Z}_2}$

So h, f mutually inverse isos. $\square$

Algebras

Let B be an A-module.

Then B is an A-algebra if B is also a ring with compatible multiplication,

i.e. $(\forall a \in A)(\forall b, c \in B)$

$$(*) \qquad a\cdot(bc) = (a\cdot b)c \qquad \left[= b(a\cdot c) \right]$$

scalar mult'n

ring multiplication

needed in noncommutative case

Consequence:

$$b(a\cdot c) = (a\cdot c)b = a\cdot(cb) = a\cdot(bc)$$

commutativity in B

The map $A \xrightarrow{\varphi} B$ is a ring hom
$a \mapsto a\cdot 1$

~~Exercise~~:

by $(*)$ & properties of scalar mult'n

Special cases:

(1) $A = F$ field, $F \xrightarrow{\varphi} B$ is injective
$\lambda \longmapsto \lambda . 1$

since $\varphi(1) = 1 \cdot 1 = 1 \neq 0$,

so $\ker\varphi \neq F$, so $\ker\varphi = \{0\}$.

(2) $A = \mathbb{Z}$, $\mathbb{Z} \xrightarrow{\varphi} B$
$n \longmapsto n . 1$

may not be injective. If $\ker\varphi = n\mathbb{Z}$

then n = characteristic of B and B

contains a copy of $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$,

called the prime subring of B.

A-algebra homomorphisms preserve

everything in sight (i.e. both ring & module laws)

Free A-algebras on n generators

$A[t_1, \ldots, t_n] = \{$ polys in $t_1, \ldots, t_n$ with coefficients from $A \}$

E

If B is any A-algebra and

$$B = \langle b_1, \ldots, b_n \rangle$$

then $t_1 \mapsto b_1, \ldots, t_n \mapsto b_n$

extends uniquely to an A-module hom:

$$A[t_1, \ldots, t_n] \twoheadrightarrow B.$$

free object

evaluation map

Tensor products of algebras

Let B, C be A-algebras.

Want multiplication on $B \otimes_A C$?

Natural candidate:

$$\left(\sum_i b_i \otimes c_i\right)\left(\sum_j b_j' \otimes c_j'\right) \overset{\text{def}}{=} \sum_{i,j} b_i b_j' \otimes c_i c_j'$$

is it well-defined??

<u>Example</u>: $F[x]\otimes_A F[x] \cong F[x,y]$

multiplication corresponds to mult^n of polys

$$(x^i\otimes x^j)(x^k\otimes x^l) = x^{i+k}\otimes x^{j+l}$$

$$\begin{array}{ccc} ||| & ||| & ||| \\ x^iy^j \quad x^ky^l & = & x^{i+k}y^{j+l} \end{array}$$

<u>Well-definedness in general</u>:

$$\begin{tikzcd}
(\alpha,\beta) \arrow[r, mapsto] & \alpha\otimes\beta & \\
(B\otimes C)\times(B\otimes C) \arrow[r, "\phi"] \arrow[ddr, "\mu"', "\text{multiplication}"] & (B\otimes C)\otimes(B\otimes C) \arrow[d, "\Theta"] & (b\otimes c)\otimes(b'\otimes c') \arrow[d, mapsto] \\
(b\otimes c, b'\otimes c') \arrow[dddr, mapsto, bend right] & B\otimes C\otimes B\otimes C \arrow[d, "\psi"] & b\otimes c\otimes b'\otimes c' \arrow[ddl, mapsto] \\
 & B\otimes C & \\
 & & \\
 & bb'\otimes cc' &
\end{tikzcd}$$

(ψ and θ come from appropriate bilinear maps)

$\uparrow$ easy $\quad$ $\uparrow$ tricky (Exercise)

<u>Sketch of existence of θ</u> :

$$\begin{tikzcd}[column sep=huge]
B \times C \arrow[r, "F_{(b',c')}"] \arrow[dr, bend right] & B \otimes C \otimes B \otimes C \\
(b,c) \arrow[r, mapsto, "\text{bilinear}"'] & b \otimes c \otimes b' \otimes c' \\
& B \otimes C \arrow[uu, bend right, "f_{(b',c')}\ \text{hom}"']
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
B \times C \arrow[r, "f"] \arrow[dr, bend right] & \mathrm{Hom}(B \otimes C,\ B \otimes C \otimes B \otimes C) \\
(b',c') \arrow[r, mapsto, "\text{bilinear}"'] & f_{(b',c')} \\
& B \otimes C \arrow[uu, "f'"'] \qquad \beta \longmapsto f'_\beta
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
(B \otimes C) \times (B \otimes C) \arrow[r, "\circledast"] \arrow[dr, bend right] & B \otimes C \otimes B \otimes C \\
(\alpha, \beta) \arrow[r, mapsto, "\text{bilinear}"'] & f'_\beta(\alpha) \\
& (B \otimes C) \otimes (B \otimes C) \arrow[uu, bend right, "\theta"']
\end{tikzcd}$$

finally !!!

$$\theta : (b \otimes c) \otimes (b' \otimes c') \longmapsto \circledast(b \otimes c,\ b' \otimes c') = f'_{b' \otimes c'}(b \otimes c)$$

$$= f_{(b',c')}(b \otimes c) = F_{(b',c')}(b,c)$$

$$= b \otimes c \otimes b' \otimes c' . \qquad \square$$