

Correction to example from last time:

$$0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/\mathbb{Z} \rightarrow 0$$

becomes

$$\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{i \otimes 1} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{q \otimes 1} \mathbb{Z}/\mathbb{Z} \otimes \mathbb{Z} \rightarrow 0$$

$$\begin{array}{c} \parallel \\ \mathbb{Z} \end{array} \xrightarrow{0} \begin{array}{c} \parallel \\ \mathbb{Z} \end{array} \xrightarrow{\cong} \begin{array}{c} \parallel \\ \mathbb{Z} \end{array} \rightarrow 0$$

$$i \otimes 1 : \underbrace{\mathbb{Z} \otimes 1}_{\substack{\text{non zero} \\ \text{generator} \\ \text{in } \mathbb{Z} \otimes \mathbb{Z}}} \mapsto \mathbb{Z} \otimes 1 = 1 \otimes \mathbb{Z} = 1 \otimes 0 = 0 \text{ in } \mathbb{Z} \otimes \mathbb{Z}$$

injectivity not preserved

$\mathbb{Z}_2$ is another non flat module

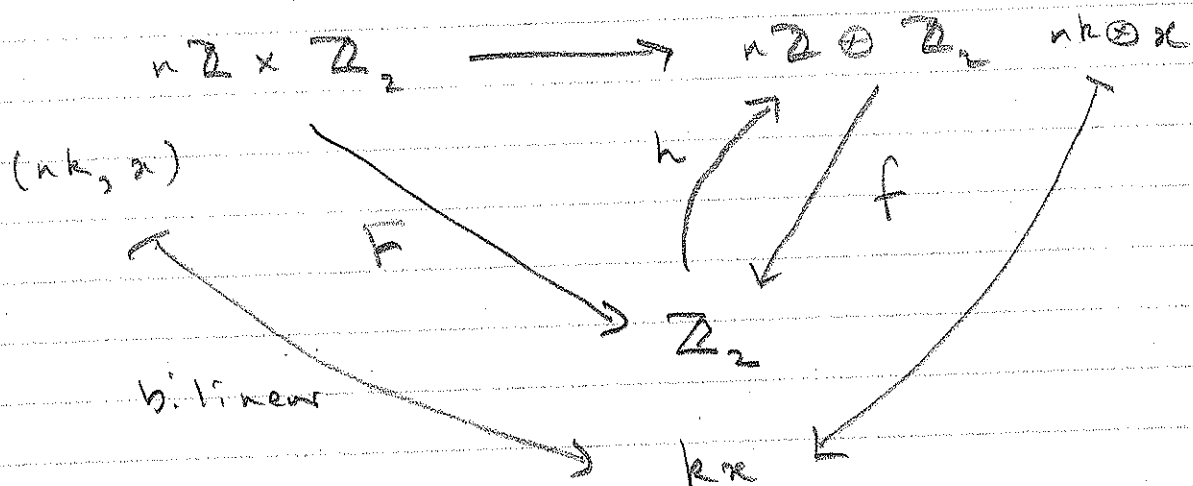
flat modules preserve all of the short exactness

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Toy "template" proof:

$$\boxed{\text{Claim: } n\mathbb{Z} \oplus_2 \mathbb{Z}_2 \cong \mathbb{Z}_2} \quad \forall n \in \mathbb{Z}^+$$

Proof:



$$\text{Let } h: \mathbb{Z}_2 \rightarrow n\mathbb{Z} \oplus_2 \mathbb{Z}_2$$

$$x \mapsto n \oplus x$$

$$\text{Then } f \circ h(x) = f(n \oplus x) = x$$

$$f \circ h = \text{id}_{\mathbb{Z}_2}$$

$$h \circ f(nk \oplus x) = h(nk)$$

$$= n \oplus nk$$

$$= k(n \oplus x) = nk \oplus x$$

$$\text{so } h \circ f \text{ fixes generators, so } h \circ f = \text{id}_{n\mathbb{Z} \oplus_2 \mathbb{Z}_2}$$

so h, f mutually inverse isom.



(c)

Algebras

Let B be an A -module.

Then B is an A -algebra if B is also a ring with compatible multiplication,

ie. $(\forall a \in A) (\forall b, c \in B)$

$$(*) \quad a \cdot (bc) = (a \cdot b) c \quad \left[= b(a \cdot c) \right]$$

↑
scalar
multⁿ

↑
ring multiplication

↑
needed in
noncommutative
case

Consequence:

$$b(a \cdot c) = (a \cdot c)b = a \cdot (cb) = a \cdot (bc)$$

↑
commutativity in B

The map $A \xrightarrow{\varphi} B$ is a ring hom
 $a \mapsto a \cdot 1$

Exercise:

by $(*)$
& properties of
scalar multⁿ

⇒

Special cases:

$$(1) \quad A = F \text{ field}, \quad \begin{array}{c} \varphi \\ F \rightarrow B \\ \lambda \mapsto \lambda \cdot 1 \end{array} \text{ is injective}$$

$$\text{since } \varphi(1) = 1 \cdot 1 = 1 \neq 0,$$

$$\text{so } \ker \varphi \neq F, \text{ so } \ker \varphi = \{0\}.$$

$$(2) \quad A = \mathbb{Z}, \quad \begin{array}{c} \varphi \\ \mathbb{Z} \rightarrow B \\ n \mapsto n \cdot 1 \end{array}$$

may not be injective, if $\ker \varphi = n\mathbb{Z}$

then $n = \underline{\text{characteristic of } B}$ and B

contains a copy of $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$,

called the prime subring of B .

A-algebra homomorphisms preserve

everything in sight (i.e. both ring & module laws)

Free A-algebras on n generators

$$A[t_1, \dots, t_n] = \{ \text{polys in } t_1, \dots, t_n \text{ with coefficients from } A \}$$

(E)

If B is any A -algebra and

$$B = \langle b_1, \dots, b_n \rangle$$

then $t_1 \mapsto b_1, \dots, t_n \mapsto b_n$

extends uniquely to an A -module hom:

$$A[t_1, \dots, t_n] \xrightarrow{\quad} B$$

free object

evaluation map

Tensor products of algebras

Let B, C be A -algebras.

Want multiplication on $B \otimes_A C$?

Natural candidate:

$$\left(\sum_i b_i \otimes c_i \right) \left(\sum_j b'_j \otimes c'_j \right) \stackrel{\text{def}}{=} \sum_{i,j} b_i b'_j \otimes c_i c'_j$$

is it well-defined??

(F)

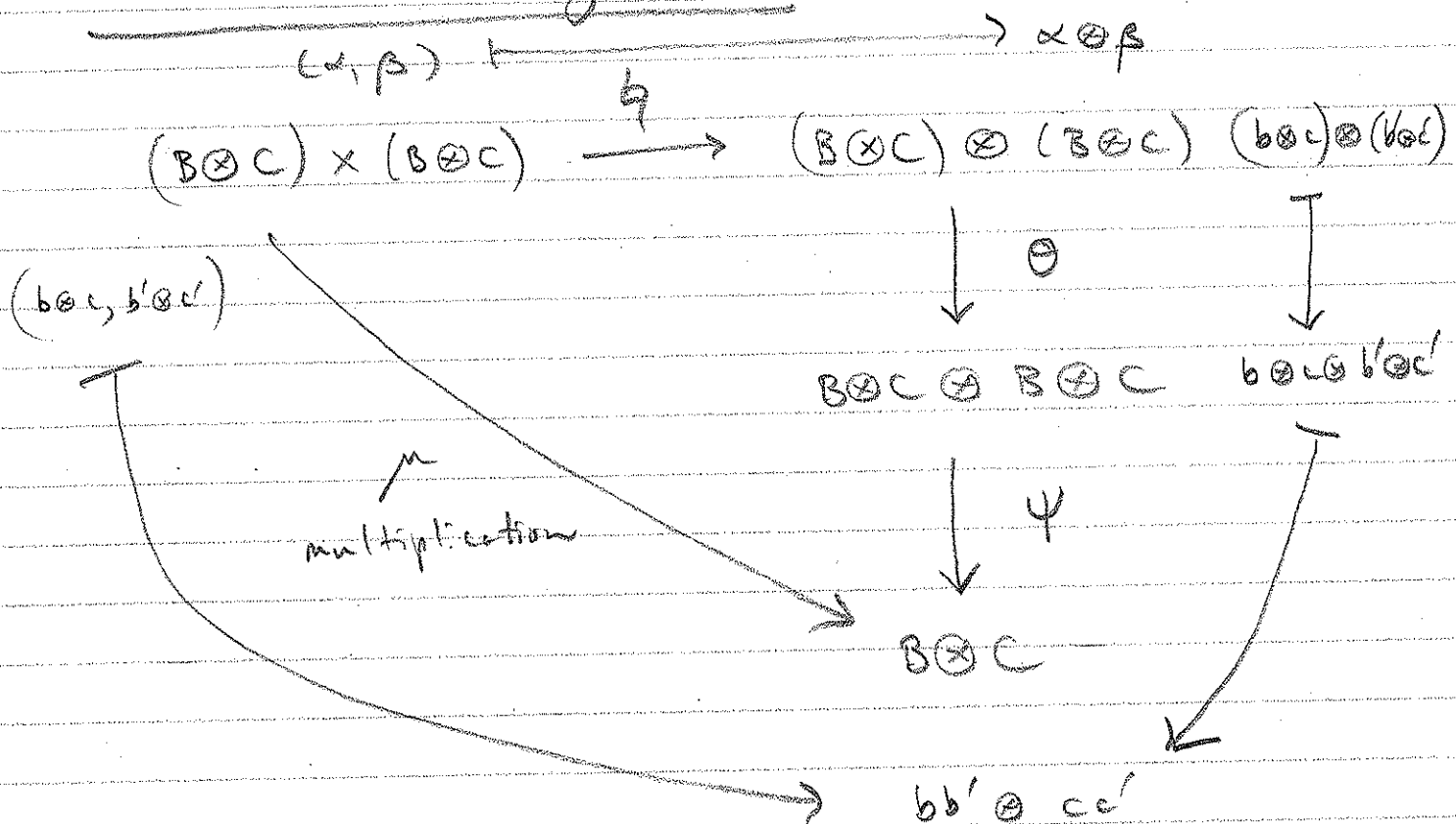
Example: $F[x] \otimes_A F[x] \cong F[x, y]$

↑
multiplication corresponds to multⁿ of polys

$$(x^i \otimes x^j)(x^k \otimes x^l) = x^{i+k} \otimes x^{j+l}$$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ x^i y^j & x^k y^l & = x^{i+k} y^{j+l} \end{array}$$

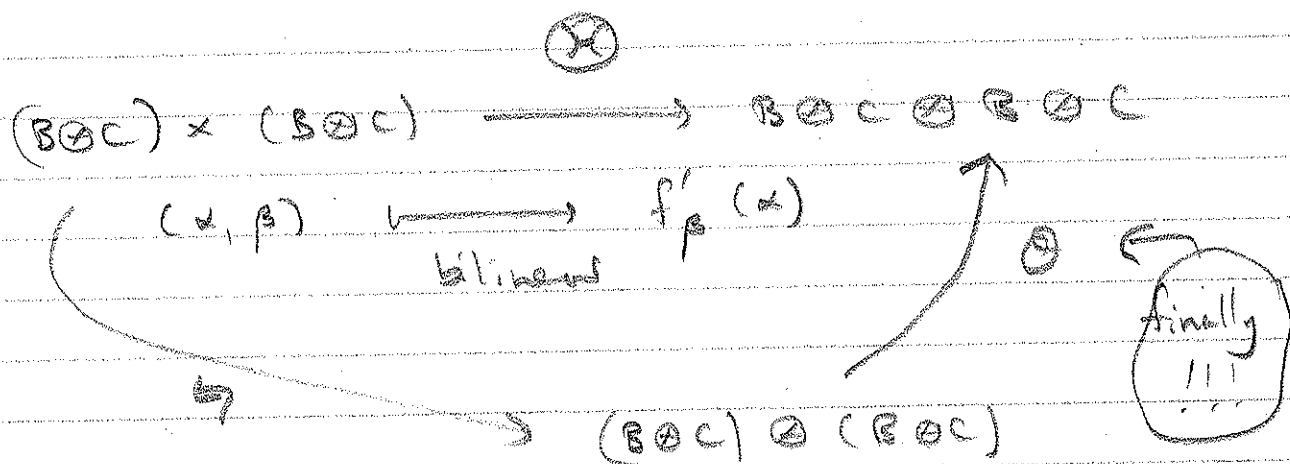
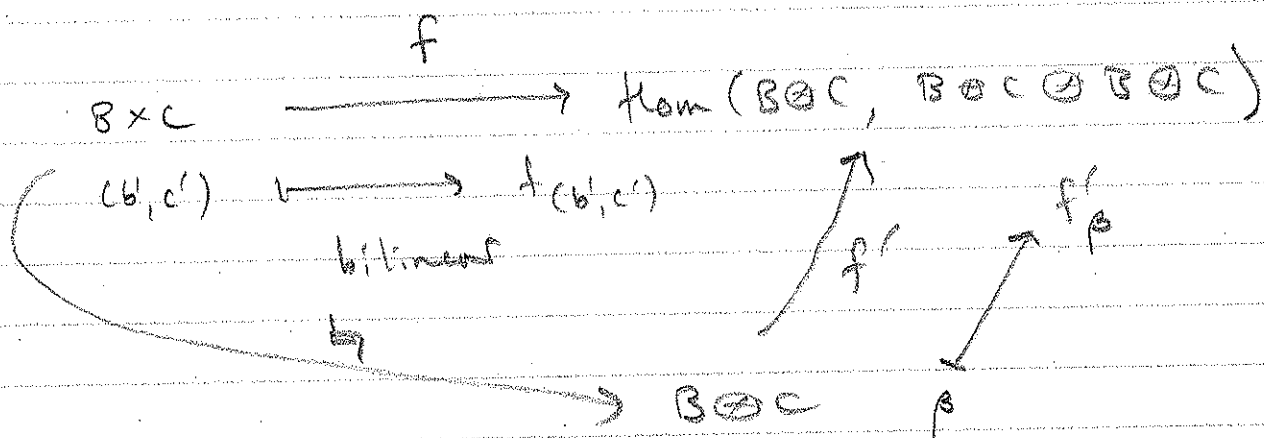
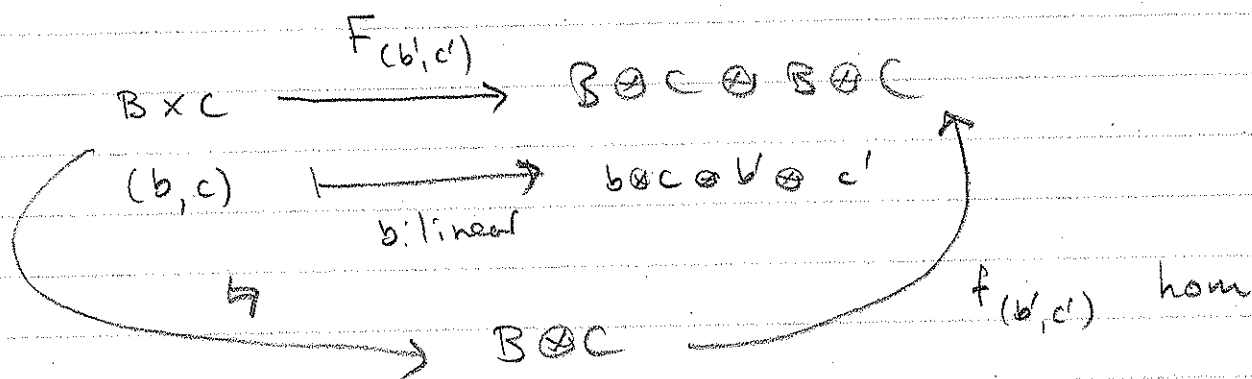
Well-definedness in general:



ψ and $\otimes$ come from appropriate bilinear maps
 ↑ easy ↑ tricky (Exercise)

⑨

Sketch of existence of θ :



$$\begin{aligned}
 \theta: (b \otimes c) \otimes (b' \otimes c') &\mapsto \bigotimes (b \otimes c, b' \otimes c') = f'_{b' \otimes c'}(b \otimes c) \\
 &= f_{(b',c')}(b \otimes c) = F_{(b',c')}(b, c) \\
 &= b \otimes c \otimes b' \otimes c'.
 \end{aligned}$$

