(A)

Commutative Algebra Lecture 26 Tues 5/5/09

Assignment Exercise: Prove

$$\boxed{(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)}$$

Getting started: want

$$\begin{tikzcd}
(M \oplus N) \times P \arrow[rr] \arrow[dr, "F"', "\text{bilinear}"] & & (M \oplus N) \otimes P \arrow[dl, bend left, "f"] \\
& (M \otimes P) \oplus (N \otimes P) \arrow[ur, bend left, "?h"] &
\end{tikzcd}$$

$$((m,n),p) \mapsto (m \otimes p, n \otimes p)$$

want this to be an iso

Look for h s.t.

$$h \circ f = 1_{(M \oplus N) \otimes P}$$

$$f \circ h = 1_{(M \otimes P) \oplus (N \otimes P)}$$

B

$$(M\otimes P)\oplus(N\otimes P) \xrightarrow{h\,?} (M\oplus N)\otimes P$$

<u>build up in stages:</u>

Find

$$\begin{tikzcd}
M\otimes P \arrow[dr, "h_1"] & \\
& (M\oplus N)\otimes P \\
N\otimes P \arrow[ur, "h_2"'] &
\end{tikzcd}$$

& glue them together

$$h(\alpha,\beta) = h_1(\alpha) + h_2(\beta)$$

from $M\otimes P$

from $N\otimes P$

addition in $(M\oplus N)\otimes P$

Restriction & extension of scalars

– moving about between rings of scalars

$$A \xrightarrow{f} B \quad \text{ring hom}$$

If N is a B-module then we can regard it as an A-module, by composing maps:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \arrow[d, "\text{scalar mult}^{\text{n}}"] \\ & \operatorname{End} N \end{tikzcd}$$

i.e. $(\forall a \in A)(\forall x \in N)$

$$ax = f(a)\,x$$

being defined

B-module scalar mult$^{\text{n}}$

& module axioms hold ✓

Classical setting : K subfield of F

$$K \xhookrightarrow{\text{inclusion}} F$$

V vector space over F, also over K

"restricting scalars to K"

Call F a *field extension* of K and write

$$F : K .$$

Put

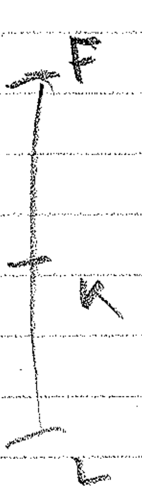
$$[F:K] = \text{dimension of } F \text{ as a vector space over } K$$

If $F:K$ and $K:L$ are field extensions then

$$\boxed{[F:L] = [F:K][F:L]}$$

Spanning half panel in printed notes

~ more general setting



E

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\text{ring mult'n}"] \arrow[d, "\text{scalar mult'n}"] & \operatorname{End} B \\ & \operatorname{End} N & \end{tikzcd}$$

N and B are modules over A

$\boxed{\begin{array}{l} \underline{\text{Prop}}: \text{ If } N \text{ is f.g. as a module over } B \\ \quad \text{and } B \text{ is f.g. as a module over } A \\ \quad \text{then } N \text{ is f.g. as a module over } A \end{array}}$

<u>Extension of scalars</u>

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr] & B \arrow[d, "\text{ring mult'n}"] \\ \operatorname{End} M & \operatorname{End} B \end{tikzcd}$$

How do we regard M as a B-module?!

no obvious composition of maps

(F)

First, regard both M and B as A-modules & form

$$M_B = B \otimes_A M$$

as an A-module.

Use B as a "buffer" to regard M_B as a B-module

$$b'(b \otimes m) \overset{\text{def}}{=} (b'b) \otimes m$$

ring mult'n

being defined on generators

then extend by linearity

Issue of well-definedness !!

— since $b \otimes m$ is an equivalence class

$((b, m) + D)$

a

For given b':

$$\begin{tikzcd}
B \times M \arrow[rr] \arrow[dr, "h"'] & & B \otimes_A M \arrow[dl, "h'"] \\
& B \otimes_A M &
\end{tikzcd}
\qquad
\begin{tikzcd}
(b,m) \arrow[dr, mapsto] & & b \otimes m \arrow[dl, mapsto] \\
& (b'b) \otimes m &
\end{tikzcd}$$

$$A \xrightarrow{f} B$$

Need to check h is bilinear with respect to A:

$$h(a_1 \cdot b_1 + a_2 \cdot b_2, x)$$

$$= b'(a_1 \cdot b_1 + a_2 \cdot b_2) \otimes x$$

$$= b'(f(a_1) b_1 + f(a_2) b_2) \otimes x$$

$$= (f(a_1) b' b_1 + f(a_2) b' b_2) \otimes x$$

$$= (a_1 \cdot (b' b_1) + a_2 \cdot (b' b_2)) \otimes x$$

$$= a_1 (b' b_1 \otimes x) + a_2 (b' b_2 \otimes x)$$

$$= a_1 h(b_1, x) + a_2 h(b_2, x)$$

✓

linearity in 2nd variable routine

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<u>Example</u>: $\mathbb{Z} \xhookrightarrow{\text{inclusion}} \mathbb{Q}$

$M = \mathbb{Z}_n$ is a $\mathbb{Z}$-module.

What is
$$M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_n$$
as a $\mathbb{Q}$-vector space?

<u>Claim</u>: $M_{\mathbb{Q}}$ is trivial

<u>Proof</u>:
$$M_{\mathbb{Q}} = \langle 1 \otimes 1 \rangle$$
and
$$1 \otimes 1 = n/n \otimes 1 = n(1/n \otimes 1)$$
$$= 1/n \otimes n.1 = 1/n \otimes 0$$
$$= 0$$
$\square$

Get something nontrivial if we try
$$\mathbb{Z} \xhookrightarrow{\text{inclusion}} A = \{x/y \in \mathbb{Q} \mid y \text{ is coprime to } n\}$$

and scalar multiplication of
$$M_A = A \otimes_{\mathbb{Z}} \mathbb{Z}_n$$
comes from
$$\frac{x}{y}(1 \otimes 1) = x \otimes y' \quad \text{where } yy' \equiv 1 \text{ mod } n.$$