Commutative Algebra   Lecture 36   Wed 27/5/09

Today we aim to prove the

Lasker – Noether Theorem: If A is a ring that is Noetherian then every proper ideal I of A has a primary decomposition.

$$I = \bigcap_{j=1}^{n} Q_j \quad \exists\, Q_1, \dots, Q_n \underset{\text{primary}}{\triangleleft} A$$

Assignment Exercise: example of a ring in which $\{0\}$ has no primary decomposition.

necessarily not Noetherian

A is Noetherian means if

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \subseteq I_n \subseteq \dots$$

is an ascending sequence of ideals then

$$(\exists N)(\forall m \geq N)\ I_m = I_N$$

called the ascending chain condition (a.c.c.)

Examples of Noetherian rings :

(1) All finite rings.

(2) $\mathbb{Z}$, $F[x]$

both PID, where "reverse division" can't go on forever, so becomes stationary

(3) $F[x_1, x_2, \ldots, x_n]$

consequence of Hilbert's Basis Theorem.

(4) $F[[x]]$

instance of last question on Assignment

Call $I \trianglelefteq A$ **irreducible** if

$(\forall J, K \trianglelefteq A) \quad I = J \cap K$

$\Rightarrow I = J$ or $I = K$.

e.g. all maximal ideals are irreducible

Pf/: $M \underset{\max}{\trianglelefteq} A$ and $M = J \cap K \Rightarrow M \subseteq J, K$

$\Rightarrow J, K \in \{M, A\} \Rightarrow J = M$ or $K = M$ (since $A \cap A = A$).

Lemma: Every ideal of a Noetherian ring is a finite intersection of irreducible ideals.

Proof (by contradiction): Suppose

$$\Sigma = \{ J \trianglelefteq A \mid J \text{ is not a finite intersection of irreducible ideals} \}$$

$$\neq \emptyset,$$

so, by a.c.c., $\exists$ maximal $M \in \Sigma$.

In particular, M is not irreducible, so

$$\boxed{M = J \cap K \quad (\exists\, J, K \trianglelefteq A), \quad M \neq J, K}$$

Hence

$$M \subsetneq J \quad \text{and} \quad M \subsetneq K,$$

so $J, K \notin \Sigma$ by maximality of M.

Hence

$$J = \bigcap \text{ irreducibles}$$

$$K = \bigcap \text{ irreducibles}$$

so

$$M = J \cap K = \bigcap \text{ irreducibles}, \text{ so } M \notin \Sigma.$$

$\times$ $\square$

Proof of Lasker–Noether Theorem:

By the Lemma it suffices to prove

$$\boxed{\text{Claim: Every proper irreducible ideal of a Noetherian ring is primary.}}$$

Proof of Claim: Suppose A is Noetherian.

If $I \underset{\neq}{\trianglelefteq} A$ is irreducible then

$$\{I\} \underset{\neq}{\trianglelefteq} A/I \text{ is irreducible,}$$

so $\boxed{\text{WLOG take } I = \{0\}}$,

Suppose $xy = 0$. WTS $x = 0$ or $y^k = 0$ $\exists k$,

But

$$\mathrm{Ann}(y) \subseteq \mathrm{Ann}(y^2) \subseteq \mathrm{Ann}(y^3) \subseteq \cdots$$

is stationary (by a.c.c.), so

$$\boxed{\mathrm{Ann}(y^n) = \mathrm{Ann}(y^{n+1}) \quad \exists n \geq 1}$$

$\boxed{\text{Subclaim: } \langle x \rangle \cap \langle y^n \rangle = \{0\}}$ $(*)$

If $z \in \langle x \rangle \cap \langle y^n \rangle$ then

$$z = vx = wy^n \quad (\exists\, v, w \in A)$$

so

$$wy^{n+1} = (wy^n)y = vxy = v0 = 0$$

so

$$w \in \operatorname{Ann}(y^{n+1}) = \operatorname{Ann}(y^n)$$

so

$$z = wy^n = 0,$$

Thus $(*)$ holds.

But $\{0\}$ is irreducible so

$$\langle x \rangle = \{0\} \text{ or } \langle y^n \rangle = \{0\}$$

so $x = 0$ or $y^n = 0$,

proving the Claim. ✓

This proves the Lasker–Noether Theorem.

$\square$