

Today we aim to prove the

Lasker-Noether Theorem: If A is a ring that is Noetherian then every proper ideal I of A has a primary decomposition.

$$I = \bigcap_{j=1}^n Q_j \quad \exists Q_1, \dots, Q_n \triangleleft A \text{ primary}$$

Assignment Exercise: example of a ring in which $\{0\}$ has no primary decomposition.

necessarily not Noetherian

A is Noetherian means if

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \subseteq I_n \subseteq \dots$$

is an ascending sequence of ideals then

$$(\exists N)(\forall m \geq N) \quad I_m = I_N$$

called the ascending chain condition (a.c.c.)

(B)

Examples of Noetherian rings:

(1) All finite rings.

(2) $\mathbb{Z}$, $F[x]$

both PID, where "reverse division"
can't go on forever, so becomes stationary

(3) $F[x_1, x_2, \dots, x_n]$

consequence of Hilbert's Basis Theorem

(4) $F[[x]]$

instance of last question on Assignment

Call $I \triangleleft A$ irreducible if

$(\forall J, K \triangleleft A) \quad I = J \cap K$

$\Rightarrow I = J \text{ or } I = K$

e.g. all maximal ideals are irreducible

Pf: $M \triangleleft A$ and $M = J \cap K \Rightarrow M \subseteq J, K$

$\Rightarrow J, K \in \{M, A\} \Rightarrow J = M \text{ or } K = M$ (since $A \cap A = A$)

(c)

Lemma: Every ideal of a Noetherian ring is a finite intersection of irreducible ideals.

Proof (by contradiction): Suppose

$$\Sigma = \{ J \triangleleft A \mid J \text{ is not a finite intersection of irreducible ideals} \} \neq \emptyset,$$

so, by a.c.c., $\exists$ maximal $M \in \Sigma$.

In particular, M is not irreducible, so

$$M = J \cap K \quad (\exists J, K \triangleleft A), \quad M \neq J, K$$

Hence

$$M \subsetneq J \quad \text{and} \quad M \subsetneq K,$$

so $J, K \notin \Sigma$ by maximality of M .

Hence

$$J = \bigcap \text{irreducibles}$$

$$K = \bigcap \text{irreducibles}$$

so

$$M = J \cap K = \bigcap \text{irreducibles}, \text{ so } M \notin \Sigma.$$



(2)

Proof of Lasker-Noether Theorem

By the lemma it suffices to prove

Claim: Every proper irreducible ideal of a Noetherian ring is primary.

Proof of Claim: Suppose A is Noetherian.

If $I \subsetneq A$ is irreducible then

$\{I\} \subsetneq A/I$ is irreducible,

so

WLOG take $I = \{0\}$

irreducible

Suppose $xy = 0$.

WTS $x = 0$ or $y^k = 0 \ \exists k$.

But

$$\text{Ann}(y) \subseteq \text{Ann}(y^2) \subseteq \text{Ann}(y^3) \subseteq \dots$$

is stationary (by a.c.c.), so

$$\text{Ann}(y^n) = \text{Ann}(y^{n+1}) \quad \exists n \geq 1$$

(E)

Sub claim: $\langle x \rangle \cap \langle y^n \rangle = \{0\}$

(*)

If $z \in \langle x \rangle \cap \langle y^n \rangle$ then

$$z = vx = wy^n \quad (\exists v, w \in A)$$

so

$$wy^{n+1} = (wy^n)y = vx y = v0 = 0$$

so

$$w \in \text{Ann}(y^{n+1}) = \text{Ann}(y^n)$$

so

$$z = wy^n = 0$$

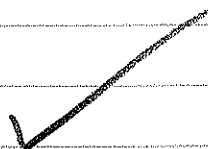
Thus (*) holds.

But $\{0\}$ is irreducible so

$$\langle x \rangle = \{0\} \text{ or } \langle y^n \rangle = \{0\}$$

$$\text{so } x=0 \text{ or } y^n=0$$

proving the claim.



This proves the Lasker-Noether theorem.

