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Commutative Algebra Lecture 32 Tue 19/5/09

Previous example reformulated:

$A = F[x]$, $S = \{1, x, x^2, x^3, \dots\}$ so

$$S^{-1}A \cong F[x, x^{-1}] = \{\text{Laurent polys}\}.$$

Put

$$M = F^{\mathbb{N}} = \{\mathbb{N}\text{-tuples}\},$$

$$M' = F^{(\mathbb{N})} = \{\underset{\sim}{\lambda} \in M \mid \underset{\sim}{\lambda} \text{ has finite support}\}$$

and let

$$\alpha : M \to M, \quad (\lambda_0, \lambda_1, \lambda_2, \dots) \mapsto (0, \lambda_0, \lambda_1, \lambda_2, \dots)$$

$$\beta : M \to M, \quad (\lambda_0, \lambda_1, \lambda_2, \dots) \mapsto (\lambda_1, \lambda_2, \lambda_3, \dots)$$

Version (α): Turn M into an $F[x]$-module by

$$p(x) \cdot \underset{\sim}{\lambda} = p(\alpha)(\underset{\sim}{\lambda})$$

linear transf. poly expression in α

and M' is an $F[x]$-submodule of M. Then

$$S^{-1}M' \cong F[x, x^{-1}] \otimes F^{(\mathbb{N})} \cong F^{(\mathbb{Z})}$$

and

$$S^{-1}M \cong F[x, x^{-1}] \otimes F^{\mathbb{N}}$$

$$\cong \{\underset{\sim}{\lambda} \in F^{\mathbb{Z}} \mid \underset{\sim}{\lambda} \text{ is eventually zero to the left}\}.$$

(B)

Version (β) : Now turn M into an $F(x)$-module

by
$$p(x) \cdot \underset{\sim}{\lambda} = p(\beta)(\underset{\sim}{\lambda}).$$

Then
$$\boxed{S^{-1}M' \cong \{\text{zero}\}}$$

Since $\dfrac{\underset{\sim}{\lambda}}{1} = \dfrac{\underset{\sim}{0}}{x^K}$ ~~$\exists$ sufficiently large K~~

since $x^K \cdot \underset{\sim}{\lambda} = (0,0,\ldots) = 1\,\underset{\sim}{0}$

and
$$S^{-1}M \cong \{\underset{\sim}{\lambda} \in F^{\mathbb{Z}} \mid \underset{\sim}{\lambda} \text{ eventually zero to the left}\} \big/ \approx$$

where $\approx$ is the equivalence relation defined by

$\underset{\sim}{\lambda} \approx \underset{\sim}{\mu}$ iff $\underset{\sim}{\lambda}$ and $\underset{\sim}{\mu}$ agree in all but finitely many positions.

As a vector space M' has basis
$$B = \{\underset{\sim}{e}_i \mid i \geq 0\}$$

where
$$\underset{\sim}{e}_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th place}}}{1}, 0, \ldots)$$

(c)

The matrix of α with respect to $\mathcal{B}$ is

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

1's below the diagonal

and of β with respect to $\mathcal{B}$ is

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

1's above the diagonal

Then $YX = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & & \ddots \end{bmatrix} = I$

identity matrix

but $XY = \begin{bmatrix} \boxed{0} & 0 & \cdots \\ 0 & 1 & \\ \vdots & & 1 \\ & & & 1 \end{bmatrix} \neq I$

Thus $\boxed{YX = I \not\Rightarrow XY = I}$

"simplest" example where this implication fails.

S^{-1} as a functor

Fix A and $S \subseteq A$, multiplicatively closed,

Then
$$S^{-1} : M \longmapsto S^{-1}M$$
becomes a functor with respect to modules and module homomorphisms. If $f : M \to N$ is an A-module hom then
$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$$
$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$
defines an $S^{-1}A$-module hom (and A-module hom by restriction of scalars) preserving diagrams:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "gf"'] & N \arrow[d, "g"] \\ & P \end{tikzcd} \quad \text{becomes} \quad \begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}f"] \arrow[dr, "S^{-1}(gf)"'] & S^{-1}N \arrow[d, "S^{-1}g"] \\ & S^{-1}P \end{tikzcd}$$
$$S^{-1}(gf) = (S^{-1}g)(S^{-1}f)$$

$$\boxed{S^{-1} \text{ preserves almost everything relating to modules}}$$

General properties :

(1) S^{-1} is exact :

$$M \xrightarrow{f} N \xrightarrow{g} P \quad \text{exact at } N$$

$$\Rightarrow \quad S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P \quad \text{exact at } S^{-1}N$$

(2) S^{-1} commutes with tensoring :

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N)$$

under $\quad \dfrac{m}{s} \otimes \dfrac{n}{t} \longmapsto \dfrac{m \otimes n}{st}$

(3) If $N, P \leq M$ then

(a) $\quad S^{-1}(N+P) = (S^{-1}N) + (S^{-1}P)$

(b) $\quad S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P)$

(c) $\quad S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$

as $S^{-1}A$-modules

(4) If M is finitely generated then

$$S^{-1}(\mathrm{Ann}(M)) = \mathrm{Ann}(S^{-1}M)$$

$$\boxed{\begin{array}{l}\underline{\text{Exercise}}: \text{ find } M \text{ where } S^{-1}(\mathrm{Ann}(M)) \subsetneq \mathrm{Ann}(S^{-1}M) \\ (\text{so } M \text{ is necessarily not finitely generated}).\end{array}}$$

__Examples:__

(i) If $M' \le M$ then $0 \to M \xhookrightarrow{i} M$ is exact, so

$0 \to S^{-1}M \xrightarrow{S^{-1}i} S^{-1}M$ is exact where

$S^{-1}i : \frac{x}{s} \mapsto \frac{x}{s}$ ← interpreted in $S^{-1}M$

interpreted in $S^{-1}M'$

so we think of $S^{-1}i$ as an injection.

(ii) Let $A = \mathbb{Z}$, $S = \{1, 2, 2^2, \dots\}$, $T = \{1, 3, 3^2, \dots\}$

$$0 \to 2\mathbb{Z} \xhookrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 0 \quad \text{exact}$$

yields

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}(2\mathbb{Z}) \arrow[r, hook, "1"] \arrow[d, equal] & S^{-1}\mathbb{Z} \arrow[r, two heads, "0"] & S^{-1}\mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
& S^{-1}\mathbb{Z} & & 0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & T^{-1}(2\mathbb{Z}) \arrow[r, hook] \arrow[d, equal] & T^{-1}\mathbb{Z} \arrow[r, two heads] & T^{-1}\mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"] & 0 \\
& 2T^{-1}(\mathbb{Z}) \arrow[r, hook] \arrow[d, equal] & T^{-1}\mathbb{Z} \arrow[r, two heads] & T^{-1}\mathbb{Z}/2T^{-1}\mathbb{Z} \arrow[d, "\cong"] & \\
& 2\left\{ \frac{z}{3^i} \,\middle|\, z \in \mathbb{Z}, i \ge 0 \right\} & & T^{-1}(\mathbb{Z}/2\mathbb{Z}) \arrow[d, "\cong"] & \\
& & & T^{-1}\mathbb{Z}_2 \cong \mathbb{Z}_2 &
\end{tikzcd}$$