A

Commutative Algebra Lecture 33 Wed 20/5/09

Example: $A = \mathbb{Z}$, $S = \{1, 2, 2^2, \ldots\}$, $T = \{1, 3, 3^2, \ldots\}$

The exact sequence

$$0 \longrightarrow 2\mathbb{Z} \xhookrightarrow{i} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

yields exact sequences:

(i)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{-1}(2\mathbb{Z}) & \xhookrightarrow{S^{-1}i} & S^{-1}\mathbb{Z} & \xrightarrow{S^{-1}q} & S^{-1}(\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & S^{-1}\mathbb{Z} & \xrightarrow{\mathbb{1}} & S^{-1}\mathbb{Z} & \xrightarrow{0} & 0 & \longrightarrow & 0 \end{array}$$

$$S^{-1}(2\mathbb{Z}) = \left\{ \frac{2z}{2^n} \;\middle|\; z \in \mathbb{Z},\ n \geq 0 \right\}$$

$$= \left\{ \frac{z}{2^m} \;\middle|\; z \in \mathbb{Z},\ m \geq 0 \right\} = S^{-1}\mathbb{Z}$$

and $S^{-1}i = \mathbb{1}$ = identity map

$S^{-1}(\mathbb{Z}/2\mathbb{Z}) = \{\text{zero}\}$ since

$$\frac{z + 2\mathbb{Z}}{1} = \frac{2\mathbb{Z}}{2} = \text{zero} \quad \forall z$$

since

$$2(z + 2\mathbb{Z}) = 2z + 2\mathbb{Z} = 2\mathbb{Z} = 1(2\mathbb{Z})$$

and $S^{-1}q : \dfrac{z}{2^m} \longmapsto \dfrac{z + 2\mathbb{Z}}{2^m} = \dfrac{z+2\mathbb{Z}}{1} \dfrac{1+2\mathbb{Z}}{2^m} = \text{zero}$

B

(ii)

$$\begin{tikzcd}
0 \arrow[r] & T^{-1}(2\mathbb{Z}) \arrow[r, hook, "T^{-1}i"] \arrow[d, equal] & T^{-1}\mathbb{Z} \arrow[r, two heads, "T^{-1}q"] \arrow[d, equal] & T^{-1}(\mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\
& 2T^{-1}\mathbb{Z} \arrow[r, hook, "i"] & T^{-1}\mathbb{Z} \arrow[r, two heads, "q"] & T^{-1}\mathbb{Z}/2T^{-1}\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
& & & \mathbb{Z}_2 &
\end{tikzcd}$$

$$\boxed{\begin{aligned} T^{-1}(2\mathbb{Z}) &= \left\{ \frac{2z}{3^n} \,\middle|\, z \in \mathbb{Z},\ n \geq 0 \right\} \\ &= 2\left\{ \frac{z}{3^n} \,\middle|\, z \in \mathbb{Z},\ n \geq 0 \right\} = 2T^{-1}\mathbb{Z} \end{aligned}}$$

$$\boxed{\begin{aligned} T^{-1}\mathbb{Z}/2T^{-1}\mathbb{Z} &= T^{-1}\mathbb{Z}/T^{-1}(2\mathbb{Z}) \\ &\cong T^{-1}(\mathbb{Z}/2\mathbb{Z}) = T^{-1}\mathbb{Z}_2 \cong \mathbb{Z}_2 \end{aligned}}$$

$$\frac{x}{3^n} = \begin{cases} \frac{0}{1} & \text{if } x = 0 \\ \frac{1}{1} & \text{if } x = 1 \end{cases}$$

since $3 = 1$ in $\mathbb{Z}_2$

If $N \subseteq M$ then

$$S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$$

$$\frac{m+N}{s} \longmapsto \frac{m}{s} + S^{-1}N$$

$$T^{-1}q : \frac{z}{3^n} \longmapsto \frac{z+2\mathbb{Z}}{3^n} \equiv \frac{z}{3^n} + 2T^{-1}\mathbb{Z}$$

(c)

Fractions & localisation

Important class of examples :

A ring, $\underset{\text{prime}}{P \trianglelefteq A}$, $S = A \setminus P$

write

$$M_P = S^{-1}M$$

and if $M \xrightarrow{f} N$ is an A-module homomorphism

write $f_P = S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$

so

$$M_P \xrightarrow{f_P} N_P$$

$$\frac{m}{s} \longmapsto \frac{f(m)}{s} \qquad (\forall\, m \in M)\,(\forall\, s \in A \setminus P).$$

Call a property $\mathcal{P}$ of modules / homomorphisms <u>local</u>

if

$$\mathcal{P} \text{ holds } \begin{cases} \forall\, M \\ \forall\, f : M \to N \end{cases}$$

$$\Longleftrightarrow \mathcal{P} \text{ holds } \begin{cases} \forall\, M_P \\ \forall\, f_P : M_P \to N_P \end{cases} \quad \left(\forall\, \underset{\text{prime}}{P \trianglelefteq A}\right)$$

Theorem: Triviality is a local property for modules, and both injectivity & surjectivity are local properties for homs.

Example: Put $M = \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6$ and let

$f : M \to M$, $(x, y, z) \longmapsto (x + 3y - 2z, 4x + 5y, x - 3z)$

Q: Is f invertible?

Put $P = 2\mathbb{Z}_6 \underset{\text{prime}}{\triangleleft} \mathbb{Z}_6$, $Q = 3\mathbb{Z}_6 \underset{\text{prime}}{\triangleleft} \mathbb{Z}_6$

so $\mathbb{Z}_6 \setminus P = \{1, 3, 5\}$, $\mathbb{Z}_6 \setminus Q = \{1, 2, 4, 5\}$

and

$M_P \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $M_Q \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

Then

$f_P : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$, $(x, y, z) \longmapsto (x + y, y, x + z)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{rank } 3$$

so f_P is invertible ✓

$f_Q : \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$, $(x,y,z) \mapsto (x+z, x+2y, x)$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{rank } 3$$

so f_Q is invertible ✓

Hence, by the Theorem, f is invertible. ✓

Theorem (triviality is local) TFAE

(i) $M = 0$.

(ii) $M_P = 0 \quad \forall P \underset{\text{prime}}{\lhd} A$

(iii) $M_Q = 0 \quad \forall Q \underset{\max}{\lhd} A$.

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) obvious.

Suppose (iii) holds and $M \neq \{0\}$, so $\exists x \in M, x \neq 0$.

But $\operatorname{Ann}(x) \underset{\neq}{\lhd} A$ (since $1 \notin \operatorname{Ann}(x)$), so

$$\operatorname{Ann}(x) \subseteq Q \underset{\max}{\lhd} A \quad \exists Q$$

(Zorn's lemma)

F

But $M_Q = \{0\}$, so

$$\frac{x}{1} = \frac{0}{1} \quad \text{in } M_Q$$

i.e. $\quad ux = u(1x - 1\,0) = 0 \quad \exists\, u \in A \setminus Q$

Hence $u \in \mathrm{Ann}(x)$ and $u \notin \mathrm{Ann}(x)$

Hence $M = \{0\}$. $\square$

Theorem (injectivity & surjectivity are local):

TFAE for a given $M \xrightarrow{\varphi} N$ module hom:

(i) φ is injective [surjective]

(ii) φ_P " " ["] $\forall\, P \trianglelefteq_{\text{prime}} A$

(iii) φ_Q " " ["] $\forall\, Q \trianglelefteq_{\max} A$.

Proof: injective half in printed notes and surjective half left as an assignment exercise.

$\square$