

Recall ring of fractions

$$S^{-1}A = \left\{ \frac{a}{s} \mid a \in A, s \in S \right\}$$

where $S \subseteq A$ multiplicatively closed, with ops

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

$$\left(\frac{a}{s} \right) \left(\frac{b}{t} \right) = \frac{ab}{st}$$

and module of fractions

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\}$$

where M is an A -module, with ops

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}$$

$$\left(\frac{a}{s} \right) \left(\frac{m}{t} \right) = \frac{am}{st}$$

becoming an $S^{-1}A$ -module.

Convention with tensor products:

Theorem: Always

$$S^{-1}M \cong S^{-1}A \otimes_A M = \left\{ \frac{1}{s} \otimes m \mid s \in S, m \in M \right\}$$

$$\frac{am}{s} \mapsto \frac{a}{s} \otimes m$$

note ordinary
brackets

(B)

$\langle \rangle$ replaced by $\{ \}$ because $\dagger$

common denominator:

$$\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^n \frac{a_i t_i}{s} \otimes m_i$$

where $s = s_1 \cdots s_n$

$t_i = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n$

$$= \sum \left(\frac{a_i t_i}{1} \frac{1}{s} \right) \otimes m_i$$

$$= \sum \frac{a_i t_i}{1} \left(\frac{1}{s} \otimes m_i \right)$$

using "buffer" scalar mult.:

$$= \sum (a_i t_i) \cdot \left(\frac{1}{s} \otimes m_i \right)$$

using restriction of scalars

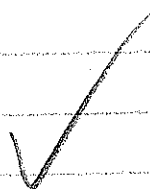
and map: $A \rightarrow s^{-1}A$

$$a \mapsto a/1$$

$$= \sum \frac{1}{s} \otimes a_i t_i m_i$$

$$= \frac{1}{s} \otimes \sum a_i t_i m_i$$

properties
of
tensor



(c)

Example: $A = F[x]$, $S = \{1, x, x^2, x^3, \dots\}$

$$M = F^{(\mathbb{N})} = \{(\lambda_0, \lambda_1, \lambda_2, \dots) \mid \lambda_i \in F, \forall i \geq 0\}$$

which is an $\mathbb{N}$ -dimensional vector space over F ,
regarded as an $F[x]$ -module, where x
corresponds to the linear transformation

$$\underline{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \dots) \mapsto (0, \lambda_0, \lambda_1, \lambda_2, \dots)$$

e.g. $(x^2 - 2x + 3) \cdot (\lambda_0, \lambda_1, \lambda_2, \dots)$

$$= x^2 \cdot \underline{\lambda} - 2x \cdot \underline{\lambda} + 3 \underline{\lambda}$$

$$= (3\lambda_0, 3\lambda_1, -2\lambda_0, \lambda_0 - 2\lambda_1 + 3\lambda_2, \lambda_1 - 2\lambda_2 + 3\lambda_3, \dots)$$

Recall

$$S^{-1}A \cong F[x, x^{-1}]$$

= ring of Laurent polys.

What is $S^{-1}M$?

$$S^{-1}M \cong S^{-1}A \otimes_A M \cong F[x, x^{-1}] \otimes_{F[x]} M$$

$$= \left\{ \frac{1}{x^n} \otimes \underline{\lambda} \mid n \geq 0, \underline{\lambda} \in M \right\}$$

(D)

Claim: $F[x, x^{-1}] \otimes_{F[x]} M \cong F(\mathbb{Z})$

where

$$x \cdot (\lambda_i)_{i \in \mathbb{Z}} \mapsto (\lambda_{i-1})_{i \in \mathbb{Z}} \quad \text{"right" translation}$$

and

$$\frac{1}{x} \cdot (\lambda_i)_{i \in \mathbb{Z}} \mapsto (\lambda_{i+1})_{i \in \mathbb{Z}} \quad \text{"left" translation}$$

Proof: Easy to check the following is bilinear with respect to $F[x]$:

$$F[x, x^{-1}] \times M \longrightarrow F(\mathbb{Z})$$

$$\left(\frac{p(x)}{x^N}, \underline{\lambda} \right) \mapsto (\dots, 0, \dots, 0, \mu_0, \mu_1, \mu_2, \dots)$$

↑
(-N)th position

where

$$p(x) \cdot \underline{\lambda} = (\mu_0, \mu_1, \mu_2, \dots),$$

which then induces a module homomorphism

$$F[x, x^{-1}] \otimes M \xrightarrow{\quad \Phi \quad} F(\mathbb{Z})$$

(E)

Put

$$e_i = (\dots, 0, 1, 0, \dots)$$

↑
ith position

for $i \in \mathbb{Z}$

$$E_j = (0, \dots, 0, 1, 0, \dots)$$

↑
jth position

for $j \in \mathbb{N}$

Then

$$\Phi : \frac{1}{2^N} \otimes E_j \mapsto e_{j-N}$$

$$\underbrace{(0, \dots, 0, 1, 0, \dots)}_{\substack{2^N \\ \text{jth position}}} \mapsto (0, \dots, 0, 1, 0, \dots, 0, \dots)_{\substack{\uparrow \\ (j-N)\text{th position}}}$$

Clearly Φ is a bijection between bases of the vector spaces (over F) $F[x, x^{-1}] \otimes M$ and $F(\mathbb{Z})$, so

$$F[x, x^{-1}] \otimes M \stackrel{\Phi}{\cong} F(\mathbb{Z})$$

with the right/left translation actions of x/x^{-1} resp. $\square$

(F)

What happens if we choose π -action

$$\pi_0(\lambda_0, \lambda_1, \lambda_2, \dots) = (\lambda_1, \lambda_2, \dots)?$$

Claim: Then $F[\pi, \pi'] \oplus M = \{\text{zero}\}$.

Proof: Let $\underline{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_k, 0, 0, \dots) \in M$

(since $\underline{\lambda}$ has finite support).

Then
$$\frac{\underline{\lambda}}{1} = \frac{0}{x^{k+1}} = \text{zero}$$

since $x^{k+1} \cdot \underline{\lambda} = (0, 0, \dots) = \underline{0} = 1 \underline{0}$.

□

What happens if we use

$$M = F^{\mathbb{N}} \quad ? \quad (\text{rather than } f(\mathbb{N}))$$

↑
unrestricted $\mathbb{N}$ -tuples

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(i) Case where $\pi \circ (\lambda_0, \lambda_1, \lambda_2, \dots) = (0, \lambda_0, \lambda_1, \lambda_2, \dots)$

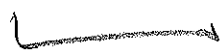
Similar to before:

$$F[\alpha, n'] \otimes M = \bigcup_{i \geq 0} F \{n \mid n \geq -i\}$$

"direct union"

where $(\lambda_{-i}, \lambda_{-i+1}, \lambda_{-i+2}, \dots)$

$\equiv (0, \dots, 0, \lambda_{-i}, \lambda_{-i+1}, \lambda_{-i+2}, \dots)$



add extra "padding" of zeros to

short sequence further back

and again ~

π induces right translation one step

$\frac{1}{\pi}$

"

left

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"

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