A

Commutative Algebra Lecture 31 Mon 18/5/09

Recall ring of fractions

$$S^{-1}A = \{ a/s \mid a \in A, s \in S \}$$

where $S \subseteq A$ multiplicatively closed, with ops

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

$$\left(\frac{a}{s}\right)\left(\frac{b}{t}\right) = \frac{ab}{st}$$

and module of fractions

$$S^{-1}M = \{ m/s \mid m \in M, s \in S \}$$

where M is an A-module, with ops

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}$$

$$\left(\frac{a}{s}\right)\left(\frac{m}{t}\right) = \frac{am}{st}$$

becoming an $S^{-1}A$-module.

Connection with tensor products:

Theorem: Always

$$S^{-1}M \cong S^{-1}A \otimes_A M = \left\{ \frac{1}{s} \otimes m \;\middle|\; s \in S, m \in M \right\}$$

$$\frac{am}{s} \longmapsto \frac{a}{s} \otimes m$$

note ordinary brackets

B

$\langle\ \rangle$ replaced by $\{\ \}$ because of common denominators:

$$\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^{n} \frac{a_i t_i}{s} \otimes m_i$$

where $s = s_1 \cdots s_n$

$t_i = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n$

$$= \sum \left(\frac{a_i t_i}{1} \frac{1}{s} \right) \otimes m_i$$

$$= \sum \frac{a_i t_i}{1} \left(\frac{1}{s} \otimes m_i \right)$$

using "buffer" scalar mult'n

$$= \sum (a_i t_i) \cdot \left(\frac{1}{s} \otimes m_i \right)$$

using restriction of scalars

and map : $A \longrightarrow S^{-1}A$

$a \longmapsto a/1$

$$\left.\begin{aligned} &= \sum 1/s \otimes a_i t_i m_i \\ &= \frac{1}{s} \otimes \sum a_i t_i m_i \end{aligned}\right\} \text{properties of tensors}$$

<u>Example</u> : $A = F[x]$, $S = \{1, x, x^2, x^3, \dots\}$

$$M = F^{(\mathbb{N})} = \{(\lambda_0, \lambda_1, \lambda_2, \dots) \mid \lambda_i \in F \ \forall i \geq 0\}$$

which is an $\mathbb{N}$-dimensional vector space over F, regarded as an $F[x]$-module, where x corresponds to the linear transformation

$$\underset{\sim}{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \dots) \longmapsto (0, \lambda_0, \lambda_1, \lambda_2, \dots)$$

e.g. $(x^2 - 2x + 3) \cdot (\lambda_0, \lambda_1, \lambda_2, \dots)$

$$= x^2 \cdot \underset{\sim}{\lambda} - 2x \cdot \underset{\sim}{\lambda} + 3\underset{\sim}{\lambda}$$

$$= (3\lambda_0, 3\lambda_1 - 2\lambda_0, \lambda_0 - 2\lambda_1 + 3\lambda_2, \lambda_1 - 2\lambda_2 + 3\lambda_3, \dots)$$

<u>Recall</u>

$$S^{-1}A \cong F[x, x^{-1}]$$

$$= \text{ring of Laurent polys.}$$

What is $S^{-1}M$?

$$S^{-1}M \cong S^{-1}A \otimes_A M \cong F[x, x^{-1}] \otimes_{F[x]} M$$

$$= \left\{ \frac{1}{x^N} \otimes \underset{\sim}{\lambda} \;\middle|\; N \geq 0, \ \underset{\sim}{\lambda} \in M \right\}.$$

Claim: $F[x,x^{-1}] \otimes_{F[x]} M \cong F^{(\mathbb{Z})}$

where
$$x \cdot (\lambda_i)_{i\in\mathbb{Z}} \mapsto (\lambda_{i-1})_{i\in\mathbb{Z}} \quad \text{"right" translation}$$

and
$$\tfrac{1}{x} \cdot (\lambda_i)_{i\in\mathbb{Z}} \mapsto (\lambda_{i+1})_{i\in\mathbb{Z}} \quad \text{"left" translation.}$$

Proof: Easy to check the following is bilinear with respect to $F[x]$:

$$F[x,x^{-1}] \times M \longrightarrow F^{(\mathbb{Z})}$$

$$\left(\frac{p(x)}{x^N}, \underset{\sim}{\lambda} \right) \longmapsto (\ldots, 0, \ldots, 0, \mu_0, \mu_1, \mu_2, \ldots)$$

$(-N)$th position

where
$$p(x) \cdot \underline{\lambda} = (\mu_0, \mu_1, \mu_2, \ldots),$$

which then induces a module homomorphism

$$F[x,x^{-1}] \otimes M \xrightarrow{\Phi} F^{(\mathbb{Z})}$$

Put

$$e_i = (\dots, 0, \underset{\substack{\uparrow \\ i\text{th position}}}{1}, 0, \dots) \qquad \text{for } i \in \mathbb{Z}$$

$$E_j = (0, \dots, 0, \underset{\substack{\uparrow \\ j\text{th position}}}{1}, 0, \dots) \qquad \text{for } j \in \mathbb{N}$$

Then

$$\Phi : \frac{1}{x^N} \otimes E_j \longmapsto e_{j-N}$$

$$\frac{(0, \dots, 0, \overset{\substack{j\text{th position} \\ \downarrow}}{1}, 0, \dots)}{x^N} \longmapsto (0, \dots, 0, \underset{\substack{\uparrow \\ (j-N)\text{th position}}}{1}, 0, \dots, 0, \dots)$$

Clearly Φ is a bijection between bases of the vector spaces (over F) $F[x, x^{-1}] \otimes M$ and $F^{(\mathbb{Z})}$, so

$$F[x, x^{-1}] \otimes M \overset{\Phi}{\cong} F^{(\mathbb{Z})}$$

with the right/left translation actions of x / x^{-1} resp. $\square$

(F)

What happens if we choose x-action

$$x \circ (\lambda_0, \lambda_1, \lambda_2, \ldots) = (\lambda_1, \lambda_2, \ldots) \; ?$$

$\boxed{\underline{\text{Claim}}:\ \text{Then}\quad F[x,x^{-1}] \otimes M = \{\text{zero}\}.}$

Proof: Let $\underset{\sim}{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_k, 0, 0, \ldots) \in M$

(since $\underset{\sim}{\lambda}$ has finite support).

Then $$\frac{\underset{\sim}{\lambda}}{1} = \frac{\underset{\sim}{0}}{x^{k+1}} = \text{zero}$$

since $x^{k+1} \cdot \underset{\sim}{\lambda} = (0, 0, \ldots) = \underset{\sim}{0} = 1\underset{\sim}{0}$. $\square$

What happens if we use

$$M = F^{\mathbb{N}} \; ? \quad (\text{rather than } F^{(\mathbb{N})})$$

unrestricted $\mathbb{N}$-tuples

(i) Case where $x \cdot (\lambda_0, \lambda_1, \lambda_2, \dots) = (0, \lambda_0, \lambda_1, \lambda_2, \dots)$

Similar to before:

$$F[x, x^{-1}] \otimes M = \bigcup_{i \geq 0} F^{\{n \mid n \geq -i\}}$$

"direct union"

where $(\lambda_{-i}, \lambda_{-i+1}, \lambda_{-i+2}, \dots)$

$$\equiv (\underbrace{0, \dots, 0,}_{} \lambda_{-i}, \lambda_{-i+1}, \lambda_{-i+2}, \dots)$$

add extra "padding" of zeros to start sequence further back

and again

x induces right translation one step

$\frac{1}{x}$ " left " " ".

(4)

(ii) Case where $x \cdot (\lambda_0, \lambda_1, \ldots) = (\lambda_1, \lambda_2, \ldots)$

<u>More complicated</u>.

$$F[x, x^{-1}] \otimes M = \bigcup_{i \geq 0} F^{\{n \mid n \geq -i\}} \Big/ \approx$$

where $\approx$ is the equivalence

$\underset{\sim}{\lambda} \approx \underset{\sim}{\mu}$ iff $\underset{\sim}{\lambda} - \underset{\sim}{\mu}$ has finite support

and now

x induces left translation one step

$\frac{1}{x}$ " right " " "

and sequences are regarded as the same if they agree in all but finitely many places.