Ⓐ

Commutative Algebra Lecture 38 Tue 2/6/09

More background on chain conditions

Throughout M is an A-module.

Call M Noetherian [Artinian] if it satisfies the a.c.c. (ascending chain condition)

[d.c.c. (descending " ")],

i.e.,

$$M_1 \leq M_2 \leq \cdots \leq M_n \leq \cdots$$

$$[\, M_1 \geq M_2 \geq \cdots \geq M_n \geq \cdots \,]$$

always stabilises, that is,

$$(\exists N)(\forall k \geq N) \quad M_k = M_N .$$

Call a ring A Noetherian [Artinian] if it is with respect to ideals (regarded as submodules).

Examples: (1) Finite abelian groups are both Noetherian and Artinian as $\mathbb{Z}$-modules.

(2) $\mathbb{Z}$ and $F[x]$ are Noetherian but not Artinian.

(3) $\mathbb{Z}$ subgroup of $(\mathbb{Q}, +)$ and

$$\mathbb{Q}/\mathbb{Z} = \{ q + \mathbb{Z} \mid q \in \mathbb{Q} \}$$

is an additive abelian group ($\mathbb{Z}$-module).

For $i \geq 0$, put

$$G_i = \left\{ \frac{a}{2^i} + \mathbb{Z} \;\middle|\; a \in \mathbb{Z} \right\}$$

and

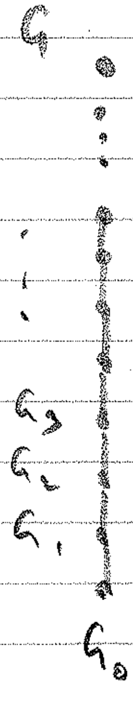
$$\boxed{G = \bigcup_{i=0}^{\infty} G_i}$$

Then

$$\begin{array}{ccccccccc} G_0 & \subsetneqq & G_1 & \subsetneqq & G_2 & \subsetneqq & G_3 & \subsetneqq & \cdots \\ 1 & & \frac{1}{2} & & \frac{1}{4} & & \frac{1}{8} & & \text{etc} \end{array}$$

is an infinite ascending chain, so G is not Noetherian as a $\mathbb{Z}$-module, but G is Artinian.

$\mathcal{L}(G)$:



Easy exercise: these are all the subgroups of G

(4) Let $H = \left\{ \frac{a}{2^n} \;\middle|\; n \geq 0,\ a \in \mathbb{Z} \right\}$

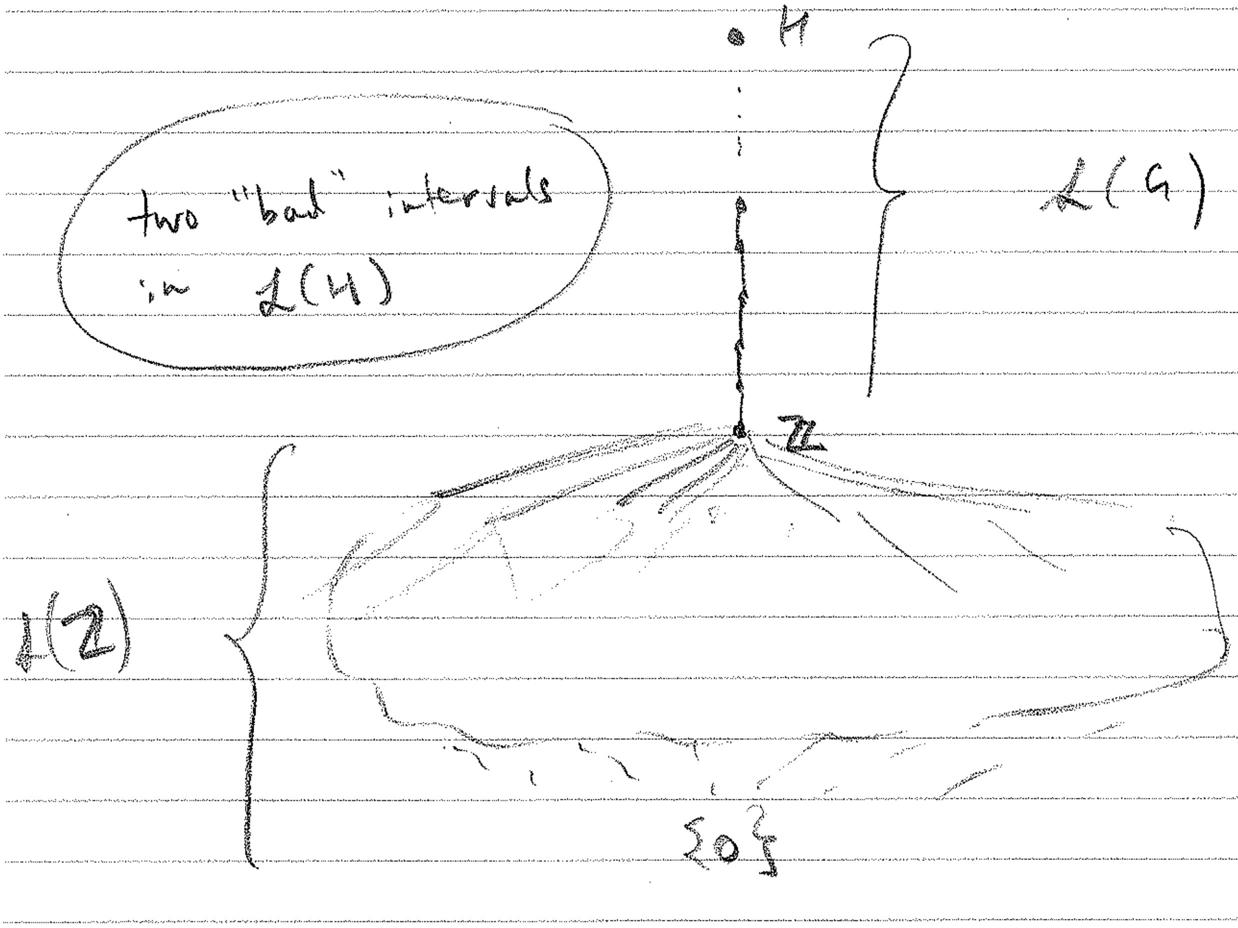
so $G = H/\mathbb{Z}$ (from previous example)

and have

$$0 \longrightarrow \mathbb{Z} \overset{i}{\hookrightarrow} H \overset{q}{\twoheadrightarrow} H/\mathbb{Z} = G \longrightarrow 0$$

exact sequence. Then

$$\boxed{H \text{ is neither } \underbrace{\text{Noetherian}}_{\text{because of } G} \text{ nor } \underbrace{\text{Artinian}}_{\text{because of } \mathbb{Z}}.}$$



(5) $F[x_1, x_2, \dots]$ not Noetherian because of

$$\langle x_1 \rangle \subsetneqq \langle x_1, x_2 \rangle \subsetneqq \dots \subsetneqq \langle x_1, \dots, x_n \rangle \subsetneqq \dots$$

and not Artinian because $F[x_1]$ isn't.

General Observation (I): Let M be an A-module. Then M is Noetherian iff every submodule is finitely generated.

Proof: ($\Rightarrow$) Easy.

($\Leftarrow$) Suppose every submodule is f.g. and

$$M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots \quad (*)$$

Put

$$M' = \bigcup_{i=1}^{\infty} M_i \leq M$$

so $\boxed{M' = \langle x_1, \dots, x_r \rangle}$ $\exists x_1, \dots, x_r \in M$.

Then

$$(\exists N) \quad x_1, \dots, x_r \in M_N,$$

so $M' \subseteq M_N \subseteq M'$, so $M' = M_N$ and $(*)$ stabilizes. $\square$

<u>General Observation 2</u> : Let

$$0 \longrightarrow M' \overset{\alpha}{\hookrightarrow} M \overset{\beta}{\twoheadrightarrow} M'' \longrightarrow 0$$

be exact. Then M is Noetherian iff both M' and M'' are Noetherian.

<u>Proof</u>: ($\Rightarrow$) Easy.

($\Leftarrow$) Suppose both M' and M'' are Noetherian and

$$L_1 \subseteq L_2 \subseteq \ldots \subseteq L_n \subseteq \ldots \qquad (*)$$

in M.

WTS $(*)$ stabilises.

Then

$$\alpha^{-1}(L_1) \subseteq \alpha^{-1}(L_2) \subseteq \ldots \subseteq \alpha^{-1}(L_n) \subseteq \ldots$$

in M'

and

$$\beta(L_1) \subseteq \beta(L_2) \subseteq \ldots \subseteq \beta(L_n) \subseteq \ldots$$

in M''

both of which stabilise, so

$$(\exists N)(\forall k \geq N) \quad \alpha^{-1}(L_k) = \alpha^{-1}(L_N), \; \beta(L_k) = \beta(L_N).$$

(F)

$$\boxed{\text{Claim: } L_k = L_N \quad \forall k \geq N}$$

Suffices to show $L_k \subseteq L_N$, so let $x \in L_k$.

Then
$$\beta(x) \in \beta(L_k) = \beta(L_N)$$
so
$$\beta(x) = \beta(y) \quad \exists\, y \in L_N$$
so
$$x - y \in \ker\beta = \operatorname{im}\alpha$$

by exactness

and
$$x - y \in L_k \quad (\text{since } L_N \subseteq L_k)$$

so $\quad x - y = \alpha(z) \quad \exists\, z \in \alpha^{-1}(L_k) = \alpha^{-1}(L_N)$

so
$$\boxed{x - y \in L_N}$$

Hence
$$x = \underbrace{x - y}_{L_N} + \underbrace{y}_{L_N} \in L_N.$$

Thus $L_k \subseteq L_N \subseteq L_k$, so $L_k = L_N$,

so the Claim is proved & (*) stabilises. $\square$

<u>Corollary ③</u>: Homomorphic images of Noetherian rings are Noetherian.

<u>Proof</u>: If $I \trianglelefteq A$ then

$$0 \longrightarrow I \hookrightarrow A \twoheadrightarrow A/I \longrightarrow 0$$

is exact. $\square$

<u>Corollary ④</u>: If $M_1, \dots, M_n$ are Noetherian then so is $M_1 \oplus \dots \oplus M_n$.

<u>Proof</u>: $M_1 \oplus \dots \oplus M_n$ is an iterated split extension. $\square$

<u>Theorem ⑤</u>: Let A be a Noetherian ring and M a finitely generated A-module. Then M is Noetherian.

<u>Proof</u>: $M \cong \underbrace{A^n/N}_{\text{homomorphic image}}$ $(\exists n)(\exists N \leq A^n)$

$A \oplus \dots \oplus A$

free module

$\square$