(A)

Commutative Algebra   Lecture 25   Mon 4/5/09

Multilinear mappings & tensor products

If $M_1, \dots, M_r, P$ are A-modules then call

$$M_1 \times \dots \times M_r \xrightarrow{f} P$$

multilinear if linear in each variable.

(fixing all the others)

Thm $\exists!\ (T, g)$ s.t.

$\forall$ multilinear $f : M_1 \times \dots \times M_r \to P$

$\exists!$ module hom $f' : T \to P$ s.t.

$$\begin{tikzcd} M_1 \times \dots \times M_r \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "f'"] \\ & P \end{tikzcd}$$

commutes.

Idea of proof: Put $C = A^{(M_1 \times \dots \times M_r)}$

free A-module

$D = \langle \qquad\qquad \rangle$

submodule

& $T = C/D$ $(= M_1 \otimes \dots \otimes M_r)$

Define

$$x_1 \otimes \cdots \otimes x_r \overset{\text{def}}{=} (x_1, \ldots, x_r) + D.$$

The generators of D are chosen so that tensor laws hold:

$(\forall i = 1, \ldots, r)$

$$\boxed{\begin{aligned} &(1) \quad x_1 \otimes \cdots \otimes x_{i-1} \otimes x \otimes x_{i+1} \otimes \cdots \otimes x_r \\ &+ \ x_1 \otimes \cdots\cdots \otimes y \otimes \cdots\cdots \otimes x_r \\ &= \ x_1 \otimes \cdots \otimes (x+y) \otimes \cdots \otimes x_r \\ &(2) \quad a(x_1 \otimes \cdots \otimes x_r) \\ &\qquad = x_1 \otimes \cdots \otimes x_{i-1} \otimes (ax_i) \otimes x_{i+1} \otimes \cdots \otimes x_r \end{aligned}}$$

Theorem (associativity):

$$\boxed{\begin{aligned} (M \otimes N) \otimes P &\cong M \otimes N \otimes P \\ &\cong M \otimes (N \otimes P). \end{aligned}}$$

(c)

Proof that $(M \otimes N) \otimes P \cong M \otimes N \otimes P$:

Build up in steps:

Fix $z \in P$ & define

$$M \times N \xrightarrow{F_z} M \otimes N \otimes P$$

$$(x, y) \longmapsto x \otimes y \otimes z$$

easily checked to be bilinear, so $\exists!\, f_z$ s.t.

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr, "F_z"'] & M \otimes N \arrow[d, "f_z"] \\ & M \otimes N \otimes P \end{tikzcd}$$

commutes.

$$\boxed{\text{Claim: } f_{az_1 + bz_2} = a f_{z_1} + b f_{z_2} \quad \forall a, b \in A,\ z_1, z_2 \in P}$$

Given this, define

$$(M \otimes N) \times P \xrightarrow{H} M \otimes N \otimes P$$

$$(\alpha, z) \longmapsto f_z(\alpha)$$

easily seen to be bilinear (by the Claim),
so $\exists!\, h$ s.t.

$$\begin{tikzcd}
(M\otimes N)\times P \arrow[r] \arrow[dr, "H"'] & (M\otimes N)\otimes P \arrow[d, "h"] \\
& M\otimes N\otimes P \arrow[u, dashed, bend left, "?\,k"]
\end{tikzcd}$$

commutes

$$\begin{aligned}
h((m\otimes n)\otimes p) &= H(m\otimes n, p) \\
&= f_p(m\otimes n) \\
&= F_p(m,n) \\
&= m\otimes n\otimes p .
\end{aligned}$$

$\boxed{\text{WTS } h \text{ is an iso, so look for an inverse?}}$

Define
$$K : M\times N\times P \longrightarrow (M\otimes N)\otimes P$$
$$(m,n,p) \longmapsto (m\otimes n)\otimes p$$
easily checked to be trilinear so $\exists!\, k$ s.t.

E

$$\begin{tikzcd}
M \times N \times P \arrow[rr] \arrow[dr, "k"'] & & M \otimes N \otimes P \arrow[dl, "k"] \\
& (M \otimes N) \otimes P \arrow[ur, bend left, "h"] &
\end{tikzcd}$$

commutes

Check: $h \circ k = 1_{M \otimes N \otimes P}$, $k \circ h = 1_{(M \otimes N) \otimes P}$

$\square$

Details of

$$\boxed{f_{az_1 + bz_2} = a f_{z_1} + b f_{z_2}}$$

Proof: Recall

$$\begin{tikzcd}
M \times N \arrow[rr] \arrow[dr, "F_z"'] & & M \otimes N \arrow[dl, "f_z"] \\
& M \otimes N \otimes P &
\end{tikzcd}$$

commutes

$$(m, n) \longmapsto (m \otimes n) \otimes z$$

Have

$$\begin{tikzcd}
M \otimes N \arrow[r, bend left, "f_{az_1 + bz_2}"] \arrow[r, bend right, "af_{z_1} + bf_{z_2}"'] & M \otimes N \otimes P
\end{tikzcd}$$

Suffices to check these agree on generators

$$m \otimes n \quad \forall m \in M,\ n \in N.$$

But

$$f_{az_1+bz_2}(m \otimes n) = F_{az_1+bz_2}(m,n)$$

$$= m \otimes n \otimes (az_1 + bz_2)$$

$$= m \otimes n \otimes (az_1) + m \otimes n \otimes (bz_2)$$

$$= a(m \otimes n \otimes z_1) + b(m \otimes n \otimes z_2)$$

$$= aF_{z_1}(m,n) + bF_{z_2}(m,n)$$

$$= af_{z_1}(m \otimes n) + bf_{z_2}(m \otimes n)$$

$$= \underbrace{(af_{z_1} + bf_{z_2})}_{\text{pointwise ops}}(m \otimes n)$$

so $f_{az_1+bz_2} = af_{z_1} + bf_{z_2}$ $\square$

Assignment Exercise: Prove

$$(M\oplus N)\otimes P \cong (M\otimes P)\oplus(N\otimes P)$$

Getting started: want

$$\begin{tikzcd}
(M\oplus N)\times P \arrow[rr] \arrow[dr, "F \text{ bilinear}"'] & & (M\oplus N)\otimes P \arrow[dl, bend left, "f"] \\
& (M\otimes P)\oplus(N\otimes P) \arrow[ur, bend left, "?h"] &
\end{tikzcd}$$

$$((m,n),p) \longmapsto (m\otimes p,\ n\otimes p)$$

want this to be an iso

Look for h s.t.

$$h\circ f = 1_{(M\oplus N)\otimes P}$$

$$f\circ h = 1_{(M\otimes P)\oplus(N\otimes P)}$$

H

$$(M \otimes P) \oplus (N \otimes P) \xrightarrow{h\,?} (M \oplus N) \otimes P$$

build up in stages:

First

$$\begin{tikzcd}
M \otimes P \arrow[dr, "h_1\,?"] & \\
& (M \oplus N) \otimes P \\
N \otimes P \arrow[ur, "h_2\,?"'] &
\end{tikzcd}$$

& glue them together.

$$h(\alpha, \beta) = h_1(\alpha) + h_2(\beta)$$

addition in $(M \oplus N) \otimes P$