

Multilinear mappings & tensor products

If $M_1, \dots, M_r, P$ are A -modules then call

$$M_1 \times \dots \times M_r \xrightarrow{f} P$$

multilinear if linear in each variable.
(fixing all the others)

Thm $\exists!$ (T, g) s.t.

$\forall$ multilinear $f: M_1 \times \dots \times M_r \rightarrow P$

$\exists!$ module hom $f': T \rightarrow P$ s.t.

$$M_1 \times \dots \times M_r \xrightarrow{g} T$$

$$\begin{array}{ccc} & & \\ & \searrow f & \swarrow f' \\ & P & \end{array}$$

commutes.

Idea of proof: $P \mapsto C = A^{(M_1 \times \dots \times M_r)}$

$$D = \langle \dots \rangle$$

submodule

$$Q \quad T = C/D (= M_1 \oplus \dots \oplus M_r)$$

free
 A -
module

(B)

Define

$$x_1 \otimes \dots \otimes x_r \stackrel{\text{def}}{=} (x_1, \dots, x_r) + \mathcal{D}$$

The generators of $\mathcal{D}$ are chosen so that tensor laws hold:

$$(\forall i=1, \dots, r)$$

$$\begin{aligned} (1) \quad & x_1 \otimes \dots \otimes x_{i-1} \otimes x \otimes x_{i+1} \otimes \dots \otimes x_r \\ & + x_1 \otimes \dots \otimes y \otimes \dots \otimes x_r \\ & = x_1 \otimes \dots \otimes (x+y) \otimes \dots \otimes x_r \end{aligned}$$

$$\begin{aligned} (2) \quad & a(x_1 \otimes \dots \otimes x_r) \\ & = x_1 \otimes \dots \otimes x_{i-1} \otimes (ax_i) \otimes x_{i+1} \otimes \dots \otimes x_r \end{aligned}$$

Theorem (associativity):

$$\begin{aligned} (M \otimes N) \otimes P & \cong M \otimes N \otimes P \\ & \cong M \otimes (N \otimes P) \end{aligned}$$

(c)

Proof that $(M \otimes N) \otimes P \cong M \otimes N \otimes P$

Build up in steps:

Fix $z \in P$ & define

$$M \times N \xrightarrow{F_z} M \otimes N \otimes P$$

$$(x, y) \mapsto x \otimes y \otimes z$$

easily checked to be bilinear, so $\exists! f_z$ s.t.

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad} & M \otimes N \\ F_z \searrow & & \swarrow f_z \\ & M \otimes N \otimes P & \end{array} \quad \text{commutes}$$

Claim: $f_{az_1 + bz_2} = af_{z_1} + bf_{z_2}$

$$\forall a, b \in A, z_1, z_2 \in P$$

Given this, define

$$(M \otimes N) \times P \xrightarrow{H} M \otimes N \otimes P$$

$$(x, z) \mapsto f_z(x)$$

⑤

easily seen to be bilinear (by the claim),
so $\exists! h$ s.t.

$$\begin{array}{ccc}
 (M \otimes N) \times P & \xrightarrow{\quad} & (M \otimes N) \otimes P \\
 \searrow H & \nearrow ? \quad k & \swarrow h \\
 & M \otimes N \otimes P &
 \end{array}
 \quad \text{commutes}$$

$$\begin{aligned}
 h((m \otimes n) \otimes p) &= H(m \otimes n, p) \\
 &= f_p(m \otimes n) \\
 &= F_p(m, n) \\
 &= m \otimes n \otimes p.
 \end{aligned}$$

WTS h is an iso, so look for an inverse?

Define

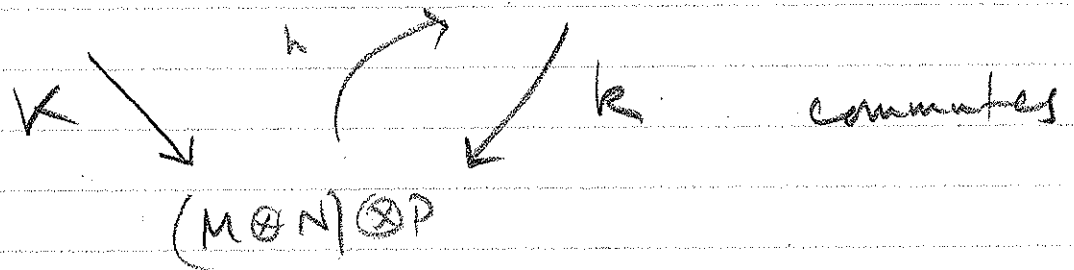
$$K: M \times N \times P \longrightarrow (M \otimes N) \otimes P$$

$$(m, n, p) \longmapsto (m \otimes n) \otimes p$$

easily checked to be bilinear so $\exists! k$ s.t.

(E)

$$M \times N \times P \longrightarrow M \otimes N \otimes P$$



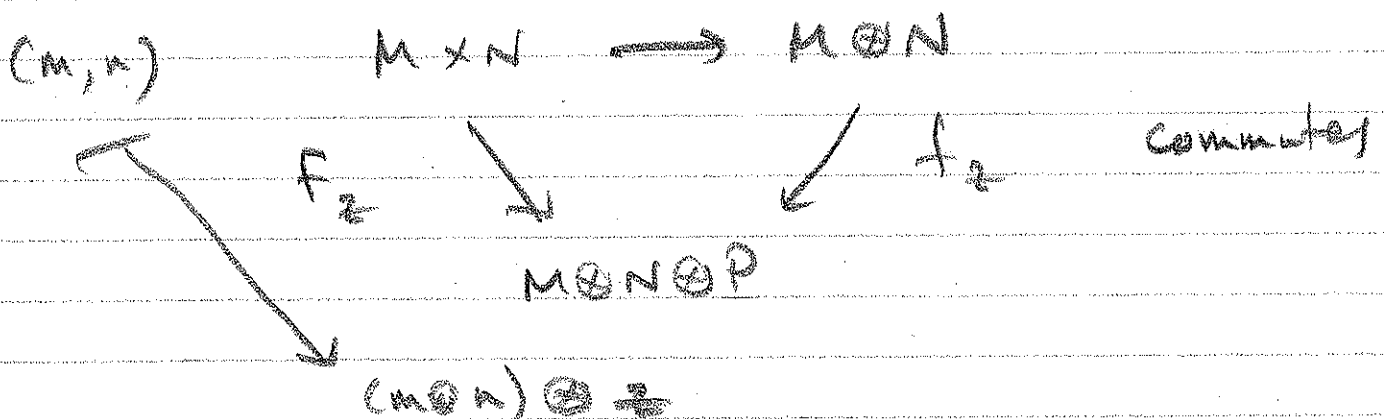
check: $h \circ k = 1_{M \otimes N \otimes P}$, $k \circ h = 1_{(M \otimes N) \otimes P}$



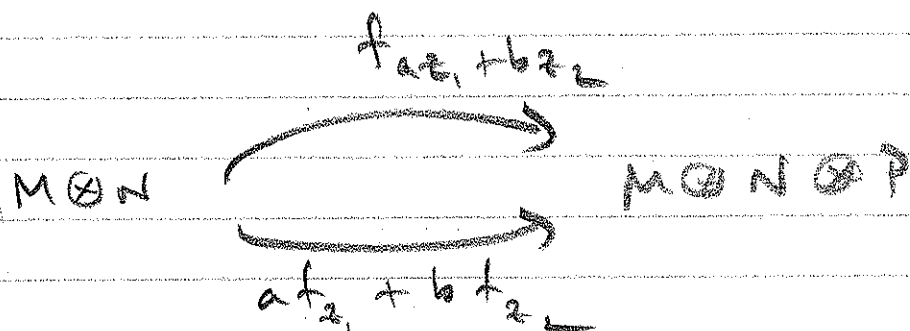
Details of

$$f_{az_1 + bz_2} = af_{z_1} + bf_{z_2}$$

Proof: Recall



Have



(F)

Suffices to check these agree on generators

$$m \oplus n \quad \forall m \in M, n \in N.$$

But

$$f_{a z_1 + b z_2}(m \oplus n) = F_{a z_1 + b z_2}(m, n)$$

$$= m \oplus n \oplus (a z_1 + b z_2)$$

$$= m \oplus n \oplus (a z_1) + m \oplus n \oplus (b z_2)$$

$$= a(m \oplus n \oplus z_1) + b(m \oplus n \oplus z_2)$$

$$= a F_{z_1}(m, n) + b F_{z_2}(m, n)$$

$$= a f_{z_1}(m \oplus n) + b f_{z_2}(m \oplus n)$$

$$= (a f_{z_1} + b f_{z_2})(m \oplus n)$$

↑
pointwise ops

so $f_{a z_1 + b z_2} = a f_{z_1} + b f_{z_2}$

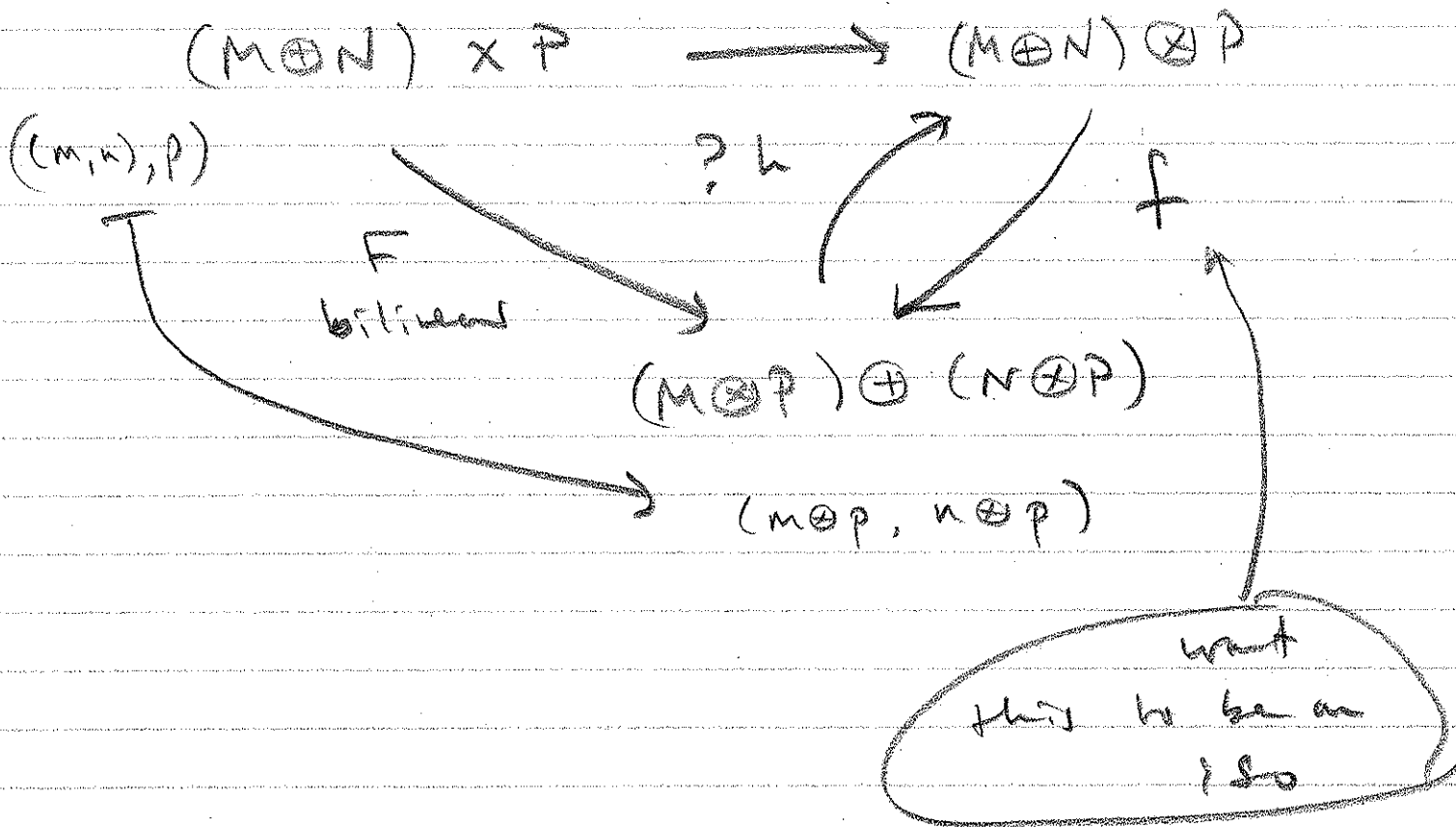


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Assignment Exercise : Done

$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$$

Getting started : want



Look for h s.t.

$$h \circ f = 1_{(M \otimes P) \oplus (N \otimes P)}$$

$$f \circ h = 1_{(M \otimes P) \oplus (N \otimes P)}$$

(H)

h ?

$$(M \otimes P) \oplus (N \otimes P) \longrightarrow (M \oplus N) \otimes P$$

build up in stages:

Find

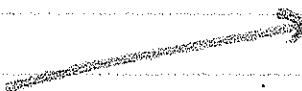
h_1 ?

$$M \otimes P$$



$$(M \oplus N) \otimes P$$

$$N \otimes P$$



h_2 ?

2 glue them together.

$$h(\alpha, \beta) = h_1(\alpha) + h_2(\beta)$$

