Semester 1 **Selected Exercises continued** 2009

Due Friday 26 June 2009. Please hand in written answers for credit. This assignment should be regarded as a take-home examination and should be done independently and without assistance from others. Acknowledge any sources, such as books or the internet, in the usual scholarly way. Throughout, ring means commutative ring with identity.

Reasonable attempts at about six questions comprise a first class effort. Anyone that successfully completes question 30 (without assistance from books or the internet) will receive a certificate of congratulation and a Mars Bar.

21. Prove that, for any integers m and $n \geq 2$,

$$\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \;\cong\; \mathbb{Z}_d$$

as $\mathbb{Z}$-algebras, where d is the greatest common divisor of m and n.

[Hint: You may use the fact, without proof, that $d = am + bn$ for some integers a and b. Verify that d annihilates the left hand side and set up a bilinear mapping from $\mathbb{Z}_m \times \mathbb{Z}_n$ onto $\mathbb{Z}_d$.]

22. Let A be a ring and S a multiplicatively closed subset of A. Verify that the rule for addition

$$a/b \,+\, c/d \;=\; (ad + cb)/(bd)$$

is well-defined in the definition of the ring of fractions

$$S^{-1}A \;=\; \{\, a/b \mid a \in A\,,\; b \in S \,\}\;.$$

Here a/b denotes the equivalence class of the ordered pair (a, b) with respect to the equivalence relation defined by

$$(a, b) \;\equiv\; (c, d)$$

if and only if $(ad - cb)u = 0$ for some $u \in S$. (You do not need to verify that this is an equivalence relation.)

23. Let $\phi : M \to N$ be an A-module homomorphism, and write $\phi_P = S^{-1}\phi$ where $S = A\backslash P$ for a prime ideal P of A. Prove that the following are equivalent:

(i) ϕ is surjective;

(ii) ϕ_P is surjective for all prime ideals P;

(iii) ϕ_Q is surjective for all maximal ideals Q.

24. Let A be a nonzero ring. Recall that an A-module M is *free* if $M \cong A^{(I)}$ for some indexing set I. If M_i is an A-module for each $i \in I$ and J is an ideal of A then it is easy to verify (you do not need to do this, it is tedious) that, as A/J-modules,

$$\bigoplus_{i \in I} M_i \Big/ J\Big(\bigoplus_{i \in I} M_i\Big) \quad \cong \quad \bigoplus_{i \in I} M_i/JM_i \ .$$

(i) Suppose $n \geq 0$ and that there is an A-module homomorphism from A^n onto $A^{(I)}$. Prove that $|I| \leq n$. (Thus $A^m \cong A^n$ as A-modules iff $m = n$.)

[Hint: Let J be maximal and apply the previous observation where $M_i = A$ for each i, to reduce the problem to one about vector spaces.]

(ii) Let M be the ideal of $\mathbb{Z}[x]$ generated by 3 and x^2. Prove that M is not free as a $\mathbb{Z}[x]$-module.

25. (i) Find a nonzero ring A such that $A \cong A \oplus A$ as rings. (Such an isomorphism as A-modules would be prohibited by 24(i).)

(ii) Find a nonzero ring A and A-module M such that $A \oplus M \cong M$ as A-modules. Prove that no such example can exist where M is finitely generated. [Hint: use 24(i).]

(iii) Prove that the subring of $\mathbb{Z}[x]$ generated by $\{\, 2x^n \mid n \geq 1 \,\}$ is not a finitely generated ring. (Thus subrings of finitely generated rings need not be finitely generated.)

26. Find a ring A, a multiplicatively closed subset S of A, and an A-module M, such that

$$S^{-1}\big(\mathrm{Ann}\,(M)\big) \ \subset \ \mathrm{Ann}\,(S^{-1}M) \qquad \text{(strict set containment)} \ .$$

27. Exhibit a proper ideal I of a ring A such that I is not primary, yet, for all $x, y \in A$,

$$xy \in I \qquad \Longrightarrow \qquad (\exists m \geq 1) \quad x^m \in I \quad \text{or} \quad y^m \in I \ .$$

28. Let F be a field. Then $G = \{\text{ functions} : \mathbb{Z}^+ \to F \,\}$ is a ring with pointwise operations and identity element $\mathbf{1} : n \mapsto 1 \quad (n \in \mathbb{Z}^+)$. Let

$$B \ = \ \{\, \mathbf{x} \in G \mid \mathbf{x} \quad \text{vanishes outside a finite set} \,\}$$

and $A = \{\, \lambda\mathbf{1} + \mathbf{x} \mid \lambda \in F \,,\ \mathbf{x} \in B \,\}$. It is easy to check that A is a subring of G and B is a maximal ideal of A. (You do not need to verify these facts.)

(i) Verify that the primary ideals of A are precisely the prime ideals, and these are B and $I_j = \{\, \mathbf{x} \in A \mid \mathbf{x}(j) = 0 \,\}$ for $j \in \mathbb{Z}^+$.

(ii) Prove that $\{\, \mathbf{0} \,\}$ does not have a primary decomposition in A.

29. Show that if a ring satisfies the descending chain condition on ideals then the nilradical and Jacobson radical are equal.

30. (i) Let A be a ring, $x \in A$, I an ideal of A such that

$$I + xA \;=\; \langle\, a_1 + xb_1\,,\ \dots\,,\ a_k + xb_k\,\rangle$$

for some $a_1, \dots, a_k \in I$ and $b_1, \dots, b_k \in A$, and

$$(I : x) \;=\; \langle\, c_1\,,\ \dots\,,\ c_\ell\,\rangle$$

for some $c_1, \dots, c_\ell \in A$. Prove that

$$I \;=\; \langle\, a_1\,,\ \dots\,,\ a_k\,,\ c_1 x\,,\ \dots\,,\ c_\ell x\,\rangle\,.$$

(ii) Prove that a ring A is Noetherian if (and clearly only if) each prime ideal of A is finitely generated.

[Hint: apply Zorn's Lemma to the set Σ of ideals of A that are not finitely generated, by first checking Σ is closed under taking unions of chains. Then use (i) to prove the maximal element of Σ is prime.]

(iii) Let P be a prime ideal of $A[[x]]$, the ring of formal power series over a ring A, and put $Q = \{\, p(0) \mid p(x) \in P\,\}$. Clearly Q is an ideal of A. Prove that if Q is generated by k elements, then P is generated by ℓ elements where $\ell = k$ if $x \notin P$ and $\ell = k + 1$ if $x \in P$.

(Thus Q is finitely generated in A iff P is finitely generated in $A[[x]]$. By (ii) we quickly deduce that if A is Noetherian then so is $A[[x]]$. When A is a field this provides an example of a Noetherian ring whose Jacobson radical is different from its nilradical.)