The University of Sydney

Commutative Algebra

Semester 1	**Selected Exercises continued**	2009

Due Friday 22 May 2009. Please hand in written answers for credit. Acknowledge any sources or assistance. Throughout, ring means commutative ring with identity.

11. Find a ring A and ideals I, J and K such that

$$(I \cap J) + (I \cap K) \neq I \cap (J + K)$$

and

$$(J + K)(J \cap K) \neq JK .$$

12. Let $A = F[x_1, \ldots, x_n]$ where F is any fixed field, and put

$$M = \{ f \in A \mid \text{the constant term of } f \text{ is zero} \} .$$

Then M is generated as an ideal of A by the elements $x_1, \ldots, x_n$. Prove that M cannot by generated as an ideal of A by fewer than n elements.

13. Let F be a field and let $f : F[x, y] \to F[x, y]$ be the ring homomorphism where $p(x, y) \mapsto p(x, x)$ for all polynomials $p(x, y) \in F[x, y]$. Prove that

$$\mathcal{C} = \{ p(x)F[x] + (x - y)F[x, y] \mid p(x) \in F[x] \}$$

and

$$\mathcal{E} = \{ p(x)F[x, y] \mid p(x) \in F[x] \} ,$$

where $\mathcal{C}$ and $\mathcal{E}$ are the sets of contracted and extended ideals respectively.

14. Find an example of a ring homomorphism and ideals J, K of the codomain such that $(J + K)^c \neq J^c + K^c$.

15. Find an example of a ring homomorphism and ideals J, K of the codomain such that $(JK)^c \neq J^c K^c$.

16. Find an example of a ring homomorphism and ideals J, K of the codomain such that $(J : K)^c \neq (J^c : K^c)$.

17. Find an example of a ring homomorphism and ideals J, K of the domain such that $(J : K)^e \neq (J^e : K^e)$.

18. Let F be a field, $n \in \mathbb{Z}^+$ and M an $n \times n$ matrix over F such that the eigenvalues of M (in the algebraic closure of F) are distinct. The vector space F^n becomes an $F[x]$-module by defining, for $p(x) \in F[x]$,

$$p(x) \cdot (a_1, \ldots, a_n) \;=\; (b_1, \ldots, b_n)$$

where the following matrix equation holds:

$$p(M) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \;=\; \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} .$$

Verify that $\mathrm{Ann}\,(F^n) \;=\; \chi(x)F[x]$ where $\chi(x)$ is the characteristic polynomial of M.

$\big[$For example, taking $M \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ leads to faithful representations of the fields

$$\mathbb{C} \;\cong\; \mathbb{R}[x]/(x^2+1)\mathbb{R}[x] \quad , \quad \mathbb{Q}(\sqrt{2}) \;\cong\; \mathbb{Q}[x]/(x^2-2)\mathbb{Q}[x]$$

by 2×2 matrices with entries from $\mathbb{R}$ and $\mathbb{Q}$ respectively.$\big]$

19. Let A be a ring and $u : N' \to N$ and $v : N \to N''$ be A-module homomorphisms. Verify that the sequence

$$0 \longrightarrow N' \xrightarrow{\;u\;} N \xrightarrow{\;v\;} N''$$

is exact iff for all A-modules M, the sequence

$$0 \longrightarrow \mathrm{Hom}(M, N') \xrightarrow{\;\bar{u}\;} \mathrm{Hom}(M, N) \xrightarrow{\;\bar{v}\;} \mathrm{Hom}(M, N'')$$

is exact.

20. Let A be a ring and M, N, P be A-modules. Prove the following A-module isomorphisms:

(i) $\quad A \otimes M \;\cong\; M$;

(ii) $\quad (M \oplus N) \otimes P \;\cong\; (M \otimes P) \oplus (N \otimes P)$.