The University of Sydney

Commutative Algebra

Semester 1 **Selected Exercises** 2009

Due Wednesday 1 April 2009. Please hand in written answers for credit. Acknowledge any sources or assistance. Throughout, ring means commutative ring with identity.

1. Let I, J be ideals of a ring A. Verify the following:

(i) $r(IJ) = r(I \cap J) = r(I) \cap r(J)$.

(ii) $r(I + J) = r(r(I) + r(J))$.

(iii) If I is prime then $r(I^n) = I$ for all positive integers n.

2. Verify that in an integral domain every prime is irreducible.

3. Verify that in a unique factorization domain every irreducible is prime.

4. Let $A = \mathbb{Z} + x\mathbb{Q}[x]$, which is a subring of $\mathbb{Q}[x]$. In particular A is an integral domain.

(i) Verify that the units of A are ± 1.

(ii) Verify that x is a nonzero nonunit of A which cannot be expressed as a product of irreducibles.

5. Let $A = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$, a subring of $\mathbb{C}$, again an integral domain.

(i) Verify that the units of A are ± 1.

(ii) Verify that 3 is irreducible but not prime.

6. Let n be a positive integer and $F_1, \ldots, F_n$ be fields. Put $A = F_1 \oplus \ldots \oplus F_n$. Prove that A has exactly n maximal ideals.

7. Let A be a finite ring.

(i) Prove that if $x \in A$ then some power of x is idempotent.

(ii) Verify that if $0 \neq e = e^2 \in A$ then $1 - e$ is idempotent and cannot be a unit.

(iii) Deduce that the nilradical and Jacobson radical of A coincide.

8. Prove that if A and B are rings and I is an ideal of $A \oplus B$ then

$$I \;=\; C \oplus D \;=\; \{\,(x,y) \mid x \in C\,,\; y \in D\,\}$$

for some ideal C of A and ideal D of B.

9. If P and Q are posets, define the poset $P \oplus Q = \{(x,y) | x \in P, y \in Q\}$ with partial order defined by

$$(x,y) \;\leq\; (z,w) \qquad \text{iff} \qquad x \leq z \quad \text{and} \quad y \leq w\,.$$

From the previous question it is easy to see that if A and B are rings with lattice of ideals $\mathcal{L}(A)$ and $\mathcal{L}(B)$ respectively, then

$$\mathcal{L}(A) \oplus \mathcal{L}(B) \;\cong\; \mathcal{L}(A \oplus B)$$

(where two posets are isomorphic if there exists a bijection between them which preserves and reflects the partial orders). Given this result, draw Hasse diagrams (using just nodes and line segments) for $\mathcal{L}(A)$ for each of the following cases:

$$A \;=\; \mathbb{Z}_3\,,\;\; \mathbb{Z}_9\,,\;\; \mathbb{Z}_{27}\,,\;\; \mathbb{Z}_3 \oplus \mathbb{Z}_9\,,\;\; \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27}$$

10. Let A be a *Boolean* ring, that is, $x^2 = x$ for all $x \in A$. Verify the following:

(i) For all $x \in A$, $x + x = 0$.

(ii) Every prime ideal P of A is maximal, and $A/P \cong \mathbb{Z}_2$.

(iii) Every finitely generated ideal of A is principal.