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COUNTING OVERLATTICES FOR POLYHEDRAL COMPLEXES

SEONHEE LIM AND ANNE THOMAS

Abstract. We investigate the asymptotics of the number of "overlattices" of a cocompact lattice Γ in $\mathrm{Aut}(X)$, where X is a locally finite polyhedral complex. We use complexes of groups to prove an upper bound for general X and a lower bound for certain right-angled buildings.

1. Introduction

The group of automorphisms $\mathrm{Aut}(X)$ of a tree (or a locally finite polyhedral complex) is a locally compact group which shares many properties with rank one simple Lie groups. This analogy has motivated many works, including the study of lattices in $\mathrm{Aut}(X)$ (see [3] and references therein).

One contrast between such Lie groups and $\mathrm{Aut}(X)$ is in the covolumes of lattices. A theorem of D. A. Kazhdan and G. A. Margulis [13] says that for a given connected semisimple Lie group G, there is a positive lower bound on the set of covolumes $\mu(\Gamma\backslash G)$ of lattices in G. On the other hand, if G is the automorphism group of a locally finite regular tree, Hyman Bass and Ravi Kulkarni [2] constructed infinite strictly ascending sequences (called *towers*) of lattices

$$\Gamma_1 < \Gamma_2 < \cdots < \Gamma_i < \cdots$$

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in G; hence, the covolumes $\mu(\Gamma_i \backslash G)$ tend to zero. A question raised by Bass and Alexander Lubotzky ([3], Section 0.7) is to find the asymptotic behavior of the number $u_\Gamma(n)$ of *overlattices* of Γ of index n, that is, the number of lattices $\Gamma' \leq G$ containing Γ with $[\Gamma' : \Gamma] = n$, as a question analogous to the question of finding subgroup growth, i.e. the number of subgroups of index n of a finitely generated group [16].

The growth of $u_\Gamma(n)$ is non-trivial (that is, $u(n) > 0$ for arbitrarily large n) if there exists a tower of lattices starting with $\Gamma_1 = \Gamma$.

In this paper, we consider the asymptotics of $u_\Gamma(n)$ for Γ cocompact in the automorphism group G of a locally finite polyhedral complex X. By arguments similar to those for tree lattices [2, Theorem 6.5], for such lattices Γ, the cardinality $u_\Gamma(n)$ is finite.

Alternatively, we could also consider the asymptotics of $\alpha_\Gamma(n) = \sum_{i \leq n} u_\Gamma(n)$, which are more stable, but there is a well-known way to interpret the asymptotics of $\alpha_\Gamma(n)$ via those of $u_\Gamma(n)$ (see [16, Chapter 1], for example).

The case where X is a tree is treated by Seonhee Lim [14]. In higher dimensions, for some X, $u_\Gamma(n)$ may have trivial growth for all cocompact lattices Γ of $\mathrm{Aut}(X)$. For example, suppose X is the classical Bruhat-Tits building associated to a higher-rank semisimple group $\mathcal{G}$ over a nonarchimedean local field of characteristic 0 (for example, $\mathcal{G} = SL_3(\mathbb{Q}_p)$). Here, $\mathcal{G}$ has finite index in $\mathrm{Aut}(X)$ [22] and the covolumes of lattices in $\mathcal{G}$ are bounded away from 0 [4]; hence, $u_\Gamma(n) = 0$ for large enough n. In contrast, the existence of towers is known for certain right-angled buildings (see [9] for towers of lattices for the product of two trees and [21] for towers of lattices for much more general right-angled buildings). Note also that, as in the tree case (see [10]), for a fixed X such that $\mathrm{Aut}(X)$ admits a tower of lattices, the existence of a tower starting with a given lattice Γ depends on Γ (see [9] for examples of irreducible lattices $\Gamma < \mathrm{Aut}(X)$, where X is the product of two regular trees, for which the growth of $u_\Gamma(n)$ is trivial).

In this work we establish an upper bound on $u_\Gamma(n)$ for very general X (see Theorem 1.1), and a lower bound for some lattices in certain right-angled buildings X (see Theorem 1.2). We note, by the previous paragraph, that there cannot be a non-trivial lower bound for general Γ and general X. Our proofs use covering theory for complexes of groups, developed by Martin R. Bridson and André Haefliger in [6] and further in our article [15].

Theorem 1.1. *Let X be a simply connected, locally finite polyhedral complex and $\Gamma \leq \mathrm{Aut}(X)$ a cocompact lattice. Then there are positive constants C_0 and C_1, depending only on Γ, such that for any $n > 1$,*

$$u_\Gamma(n) \leq (C_0 n)^{C_1 \log^2(n)}.$$

This bound is asymptotically the same as the upper bound for tree lattices in [14]. Although the proof uses the same deep results of finite group theory, Bridson and Haefliger's [6] definition of covering of complexes of groups makes this bound easier to obtain than the result for trees (which uses Bass's covering theory for graphs of groups [1]), thus giving a simpler proof of the tree case. Alternatively, we can obtain this upper bound using the upper bound for tree lattices (see section 3).

The lower bound, proved in subsection 3.3, is for certain right-angled buildings. A special case of the lower bound we obtain is presented below.

Theorem 1.2. *Let q be prime and let X be a Bourdon building $I_{p,2q}$ (see [5]). Then there are a cocompact lattice Γ in $\mathrm{Aut}(X)$ and constants C_0 and C_1, such that for any $N > 0$, there exists $n > N$ with*

$$u_\Gamma(n) \geq (C_0 n)^{C_1 \log n}.$$

The full statement, in Theorem 3.2, applies to more general right-angled buildings, including examples in arbitrarily high dimension. The proof applies the Functor Theorem of [21] to a construction for tree lattices in [14].

Theorems 1.1 and 3.2, together with the examples given above for buildings, are presently the only known behaviors for overlattice counting functions in higher dimensions.

2. Background

In subsection 2.1 we describe the topology on the group G of automorphisms of a locally finite polyhedral complex X; in 2.2 we characterize cocompact lattices in G using a combinatorial normalization of Haar measure. We also recall the basic theory of complexes of groups. Subsection 2.3 states the pertinent results of [15] on covering theory for complexes of groups. Subsection 2.4 defines right-angled buildings and recalls properties that we will need to prove Theorem 3.2.

2.1. Lattices in automorphism groups of polyhedral complexes

Let X be a connected, locally finite polyhedral complex, with first barycentric subdivision X'. Let $V(X')$ be the set of vertices of X', which is in bijection with the set of cells of X. Let $\mathrm{Aut}(X)$ be the group of cellular isometries of X acting *without inversions* on X, i.e., its elements fix pointwise each cell that they preserve.

The group $G = \mathrm{Aut}(X)$ is naturally a locally compact group with the compact-open topology. In this topology, a subgroup $\Gamma \leq G$ is discrete if and only if it acts on X with finite cell stabilizers. For a given cell $s \in V(X')$, let Γ_s be its stabilizer in Γ. For a given cell $s \in \Gamma \backslash V(X')$ in a quotient complex, by abuse of notation, let us denote by $|\Gamma_{\tilde{s}}|$ the cardinality of the stabilizer of a lift $\tilde{s}$ of s in $V(X')$. Note that two stabilizers of two lifts u, v of s have the same cardinality since they are conjugate by an element of $\mathrm{Aut}(X)$ sending u to v.

Using Jean-Pierre Serre's normalization [19] and the same arguments as for tree lattices [3, Chapter 1], it can be shown that if $G \backslash X$ is finite, then a discrete subgroup $\Gamma \leq G$ is a lattice (that is, $\mu(\Gamma \backslash G) < \infty$ where μ is left-invariant Haar measure on G) if and only if its $V(X')$-covolume

$$\sum_{s \in \Gamma \backslash V(X')} \frac{1}{|\Gamma_{\tilde{s}}|},$$

which we will denote by $\mathrm{Vol}(\Gamma \backslash\backslash V(X'))$, is finite. (Here, $\Gamma \backslash\backslash V(X')$ denotes the quotient complex of groups of Γ which will be defined in subsection 2.2. See the paragraph just below the definition of a complex of groups.)

A lattice Γ is cocompact (that is, the quotient $\Gamma \backslash G$ is compact) if and only if this sum has finitely many terms. We now normalize the Haar measure μ on $G = \mathrm{Aut}(X)$ so that for all lattices $\Gamma \leq G$, the covolume of Γ is

$$\mu(\Gamma \backslash G) = \mathrm{Vol}(\Gamma \backslash\backslash V(X')).$$

2.2. Complexes of groups

We give here the definitions from Bridson and Haefliger's theory of complexes of groups [6] needed to state the relevant results of [15].

Let Y be a polyhedral complex with barycentric subdivision Y'. Let $V(Y')$ be the set of vertices of Y' and $E(Y')$ its set of edges. Each $a \in E(Y')$ corresponds to cells $\tau \subset \sigma$ of Y and so may be oriented from σ to τ, with $i(a) = \sigma$ and $t(a) = \tau$. Two edges a and b of Y' are *composable* if $i(a) = t(b)$, in which case there exists an edge $c = ab$ of Y' such that $i(c) = i(b)$, $t(c) = t(a)$ and a, b and c form the boundary of a 2-simplex in Y'.

A *complex of groups* $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ over a polyhedral complex Y is given by:

(1) a group G_σ for each $\sigma \in V(Y')$, called the *local group* at σ;
(2) a monomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$ for each $a \in E(Y')$;
(3) for each pair of composable edges a, b in Y', a "twisting" element $g_{a,b} \in G_{t(a)}$, such that

$$\mathrm{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where $\mathrm{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$, and for each triple of composable edges a, b, c, the following cocycle condition holds

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c} \, .$$

Similar to Bass-Serre theory of graphs of groups, if Γ is a lattice in the automorphism group $\mathrm{Aut}(X)$ of a connected, locally finite polyhedral complex X, then we can define a quotient complex of groups $\Gamma\backslash\backslash V(X')$, by taking the quotient complex $\Gamma\backslash V(X)$ as the base polyhedral complex Y, the stabilizer $\Gamma_{\tilde{s}}$ of any lift $\tilde{s}$ of $s \in \Gamma\backslash V(X')$ for the local group at s, and the injection followed by a suitable conjugation h_a (sending $i(a)$ to $t(a)$) for the monomorphism ψ_a for each $a \in E(Y')$, and elements $g_{a,b} = h_a h_b h_{ab}^{-1}$. We denote this complex of groups by $\Gamma\backslash\backslash V(X')$.

Let $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ and $H(Z) = (H_\tau, \psi_{a'}, g_{a',b'})$ be complexes of groups over polyhedral complexes Y and Z, respectively. Let $f : Y' \to Z'$ be a simplicial map sending vertices to vertices and edges to edges (such an f is *nondegenerate*). A *morphism* $\phi : G(Y) \to H(Z)$ over $f : Y' \to Z'$ consists of

(1) a homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in V(Y')$, and

(2) an element $\phi(a) \in H_{t(f(a))}$ for each $a \in E(Y')$, such that

$$\mathrm{Ad}(\phi(a)) \circ \psi_{f(a)} \circ \phi_{i(a)} = \phi_{t(a)} \circ \psi_a,$$

where $\mathrm{Ad}(\phi(a))(g) = \phi(a)g\phi(a)^{-1}$, and for all pairs of composable edges (a,b) in $E(Y')$,

$$\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi_{f(a)}(\phi(b))g_{f(a),f(b)}.$$

If f is an isomorphism of simplicial complexes and each ϕ_σ is an isomorphism of groups, then the morphism ϕ is an *isomorphism*.

A morphism of complexes of groups $\phi : G(Y) \to H(Z)$ is a *covering* if

(1) each ϕ_σ is injective, and
(2) for each $\sigma \in V(Y')$ and $b \in E(Z')$ such that $t(b) = f(\sigma) = \tau$, the map

$$\coprod_{a\in f^{-1}(b), t(a)=\sigma} G_\sigma/\psi_a(G_{i(a)}) \to H_\tau/\psi_b(H_{i(b)})$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection.

An isomorphism of complexes of groups is clearly a covering. From condition (2) above in the definition of covering and the connectedness of Y', if the value of

$$n := \sum_{\sigma\in\, f^{-1}(\tau)} \frac{|H_\tau|}{|G_\sigma|} = \sum_{a\in f^{-1}(b)} \frac{|H_{i(b)}|}{|G_{i(a)}|}$$

is finite, it is independent of the vertex τ and the edge b. A covering of complexes of groups with the above n is said to be *n-sheeted*.

Any action (without inversions) by a group G on a polyhedral complex X induces a complex of groups over the quotient $G\backslash X$, which is unique up to isomorphism of complexes of groups. A complex of groups is *developable* if it is isomorphic to a complex of groups induced by an action.

Bridson and Haefliger [6] proved that a certain local condition, called *nonpositive curvature*, of a complex of groups $G(Y)$ ensures developability. To define this, for some $\kappa \le 0$, each cell of Y must be equipped with a Riemannian metric of constant sectional curvature κ. For each vertex σ of Y', there is a simplicial complex called the *local development* at σ, which is defined combinatorially using the local groups of $G(Y)$ at σ and neighboring vertices. Gromov's Link Condition (see [6]) implies that if, for each $\sigma \in V(Y')$, the link of the local development at σ is CAT(1), then $G(Y)$ is nonpositively curved. In particular, if $\dim(Y) = 2$, then the link of the local development at each $\sigma \in V(Y')$ is a metric graph, and $G(Y)$ is nonpositively curved if and only if each of these links contains no circuits of length $< 2\pi$.

The *fundamental group* $\pi_1(G(Y), T)$ of a complex of groups $G(Y)$ is defined with respect to a choice of maximal tree T in the 1-skeleton of Y', so that if Y is simply connected and all twisting elements $g_{a,b}$ are trivial, then $\pi_1(G(Y), T)$ is isomorphic to the direct limit of the family of groups G_σ and monomorphisms ψ_a.

If a complex of groups $G(Y)$ is developable, then its *universal cover* $D(G(Y), T)$ is a connected, simply-connected polyhedral complex. Different choices of trees T result in isometric universal covers. The universal cover is equipped with a natural action of $\pi_1(G(Y), T)$, so that the complex of groups induced by the action of the fundamental group on the universal cover $D(G(Y), T)$ is canonically isomorphic to $G(Y)$.

Let $G(Y)$ be a developable complex of groups with fundamental group Γ and universal cover X. We say that $G(Y)$ is *faithful* if Γ acts faithfully on X, in which case Γ may be regarded as a subgroup of $\mathrm{Aut}(X)$. In this case, by subsection 2.1, Γ is a cocompact lattice in $\mathrm{Aut}(X)$ if and only if all local groups of $G(Y)$ are finite and Y is a finite polyhedral complex.

2.3. Covering theory for complexes of groups

To count overlattices, we use several results from our previous paper [15], which we recall in this subsection. The main result, Theorem 2.1, gives a one-to-one correspondence between isomorphism classes of coverings of complexes of groups and overlattices.

Theorem 2.1. *Let X be a simply connected, locally finite polyhedral complex, and let Γ be a cocompact lattice in $\mathrm{Aut}(X)$ (acting without inversions) which induces a complex of groups $G(Y)$ over $Y = \Gamma\backslash X$. Then there is a bijection between the set of overlattices of Γ of index n (acting without inversions) and the set of isomorphism classes of n-sheeted coverings of faithful developable complexes of groups by $G(Y)$.*

The definition of isomorphism of coverings is given at the end of this subsection.

We will need Proposition 2.2, which gives sufficient conditions for a developable complex of groups $G(Y)$ to be faithful. For any choice of tree T in the 1-skeleton on Y', there is a canonical morphism of complexes of groups

$$\iota_T : G(Y) \to \pi_1(G(Y), T)$$

which is injective on each local group G_σ (here, the group $\pi_1(G(Y), T)$ is considered as a complex of groups over a single vertex).

Proposition 2.2. *Let $G(Y)$ be a developable complex of groups over a connected polyhedral complex Y. Choose a maximal tree T in the 1-skeleton of Y', and identify each local group G_σ with its image in $\pi_1(G(Y), T)$ under ι_T. Let*

$$N_T = \ker(\pi_1(G(Y), T) \to D(G(Y), T)).$$

Then

1. *N_T is a vertex subgroup, that is, $N_T \leq G_\sigma$ for each $\sigma \in V(Y')$.*
2. *N_T is Y-invariant, that is, $\psi_a(N_T) = N_T$ for each $a \in E(Y')$.*
3. *N_T is normal, that is, $N_T \trianglelefteq G_\sigma$ for each $\sigma \in V(Y')$.*
4. *N_T is maximal: if N'_T is another Y-invariant normal vertex subgroup, then $N'_T \leq N_T$.*

The following result appears as Proposition 2.5 in [15], where the induced maps Λ_{T_1,T_2} and $L^\lambda_{T_1,T_2}$ are explicitly defined.

Proposition 2.3. *Let $\lambda : G(Y_1) \to G(Y_2)$ be a covering of complexes of groups over a nondegenerate simplicial map $l : Y'_1 \to Y'_2$, where Y_1 and Y_2 are connected polyhedral complexes. Assume $G(Y_1)$ and $G(Y_2)$ are developable. For any maximal trees T_1 and T_2 in the 1-skeletons of Y'_1 and Y'_2, respectively, there is an induced monomorphism of fundamental groups*

$$\Lambda_{T_1,T_2} : \pi_1(G(Y_1), T_1) \to \pi_1(G(Y_2), T_2)$$

and a Λ_{T_1,T_2}-equivariant isomorphism of universal covers

$$L^\lambda_{T_1,T_2} : D(G(Y_1), T_1) \to D(G(Y_2), T_2).$$

Proposition 2.3 is used to define an isomorphism of coverings as follows. Let $\lambda : G(Y_1) \to G(Y_2)$ and $\lambda' : G(Y_1) \to G(Y_3)$ be coverings of developable complexes of groups over connected polyhedral complexes. We say that λ and λ' are *isomorphic* coverings if, for any choice of maximal trees T_1, T_2, and T_3 in Y_1, Y_2, and Y_3, respectively, there exists an isomorphism $\lambda'' : G(Y_2) \to G(Y_3)$ of complexes of groups such that the diagram of induced isomorphisms of universal covers commutes, that is,

$$L^{\lambda''}_{T_2,T_3} \circ L^{\lambda}_{T_1,T_2} = L^{\lambda'}_{T_1,T_3}.$$

2.4. Right-angled buildings

We recall the definition and some properties of right-angled buildings for which we obtain a nontrivial lower bound in Theorem 3.2. This class of buildings contains not only right-angled hyperbolic buildings, but also some Euclidean buildings (which are not classical Bruhat-Tits buildings). We mostly follow [7].

Recall that a *Coxeter system* (W, S) is a group W with presentation

$$W = \langle\, S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \,\rangle$$

where $m_{ss} = 1$ for all $s \in S$ and $m_{st} \in \{2, 3, \ldots\} \cup \{\infty\}$ for $s \neq t$ in S, with $m_{st} = \infty$ meaning that the element st has infinite order. A Coxeter system is *right-angled* if for all $s, t \in S$ with $s \neq t$, $m_{st} \in \{2, \infty\}$.

For any Coxeter system (W, S) there is a locally finite simplicial complex Σ, called the *Davis complex* for (W, S), on which W acts properly discontinuously and cocompactly by isometries. The definition is as follows.

A subset T of S is called *spherical* if the subgroup $W_T \leq W$ generated by elements of T is a finite subgroup. The poset $\mathcal{S}_{>\phi}$ of all nonempty spherical subsets, as an abstract simplicial complex, is called the *nerve* for (W, S).

Let K be the cone on the barycentric subdivision of the nerve L for (W, S). Write $\mathcal{S}$ for the set of subsets $T \subset S$ such that the subgroup W_T of W generated by T is finite. By convention, $W_\emptyset = 1$. There is then a one-to-one correspondence between the vertices of K and the types $T \in \mathcal{S}$, with the cone point having type $\emptyset$. For each $s \in S$, let K_s be the closed star of the vertex $\{s\}$ in the barycentric subdivision of L. In other words, it is the union of simplices of L containing s. We call K_s the *s-mirror of K*. For each $x \in K$, put

$$S(x) = \{s \in S \mid x \in K_s\}.$$

Then the Davis complex Σ is defined by

$$\Sigma := (W \times K)/\sim$$

where $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. The natural W-action on $W \times K$ induces a W-action on Σ with strict fundamental domain K. This W-action preserves types, so each vertex of Σ has type some $T \in \mathcal{S}$.

The Davis complex Σ may be equipped with a standard piecewise Euclidean metric, so that it is a CAT(0) space, and conditions on (W, S) are known for when Σ may be equipped with a standard piecewise hyperbolic metric, so that it is CAT(-1) (see [7]). From now on we assume that Σ is equipped with one of these standard metrics.

Definition 2.4. Let (W, S) be a right-angled Coxeter system. A *right-angled building* of type (W, S) is a polyhedral complex X, equipped with a maximal family of subcomplexes, called *apartments*. Each apartment is polyhedrally isometric to the Davis complex Σ for (W, S), and the copies of K in X are called *chambers*. The apartments and chambers of X satisfy the axioms

(1) any two chambers of X are contained in a common apartment; and
(2) given any two apartments Σ and Σ' of X, there is an isometry $\Sigma \to \Sigma'$ which fixes the intersection $\Sigma \cap \Sigma'$.

Note that X, although a building, is not, in general, isomorphic to any classical Bruhat-Tits building for an algebraic group over a nonarchimedean local field.

Each vertex of a right-angled building X has a type $T \in \mathcal{S}$, induced by the types of vertices of its apartments, and any copy of an s-mirror K_s of a chamber K in X will be called an *s-mirror of X*. For $s \in S$, an *$\{s\}$-residue of X* is a connected subcomplex consisting of all chambers which meet a given s-mirror of X. A right-angled building X is *regular* if for each $s \in S$, there is a cardinality $q_s \geq 2$ such that every $\{s\}$-residue of X contains exactly q_s chambers. We will refer to a regular right-angled building of type (W, S) as a *building of type (W, S) and parameters $\{q_s\}_{s \in S}$*.

The following result classifies regular right-angled buildings and will be used in the proof of Theorem 3.2 below.

Theorem 2.5 (Proposition 1.2, [12])**.** *Let (W,S) be a right-angled Coxeter system and $\{q_s\}_{s\in S}$ a family of cardinalities ($q_s \geq 2$). Choose the type of standard metric on the Davis complex Σ for (W,S), either piecewise Euclidean or (if possible) piecewise hyperbolic. Then there exists a unique (up to isometry) building X of type (W,S), with the given standard metric, such that for each $s \in S$, the $\{s\}$-residue of X has cardinality q_s.*

The result in the 2-dimensional case (with hyperbolic metric) is due to M. Bourdon [5]. According to [12, p. 139], Theorem 2.5 was proved by M. Globus, and was known to M. Davis, T. Januszkiewicz, and J. Świątkowski.

For example, let (W,S) be the Coxeter system generated by reflections in the sides of a regular right-angled hyperbolic p-gon P (with $p \geq 5$). The Davis complex Σ for (W,S) may then be equipped with the standard piecewise hyperbolic metric, and with this metric is isometric to the barycentric subdivision of the tesselation of the hyperbolic plane by copies of the regular polygon P. An example of a regular building of type (W,S) is (the barycentric subdivision of) *Bourdon's building* $X = I_{p,q}$ in which every 2-cell is a copy of the p-gon P, and there are q copies of P glued around each 1-cell of X, with $q \geq 2$.

3. Proof of main results

We prove first the upper bound of Theorem 1.1 and conclude with the proof for the lower bound for right-angled buildings.

3.1. Upper bound

We now prove the upper bound of Theorem 1.1, stated in the introduction, using the bijection between overlattices and coverings given in Theorem 2.1. We will also use the following deep results of finite group theory.

Suppose G is a group of order $m = \prod_{i=1}^{t} p_i^{k_i}$ (p_i are distinct primes, i.e., k_i are maximal) and let $\mu(m) = \max\{k_i\}$. By results of Andrea Lucchini [17], Robert M. Guralnick [11], and Charles C. Sims [20], the minimal number $d(G)$ of generators of G is bounded above by $\mu(m)+1$. By work of L. Pyber [18] and Sims [20], there is also an upper bound on the number $f(m)$ of isomorphism classes of groups of order m:

$$f(m) \leq m^{g(m)},$$

where $g(m) = \frac{2}{27}\mu(m)^2 + \frac{1}{2}\mu^{5/3}(m) + 75\mu(m) + 16$.

Now let Γ be a cocompact lattice in G, the automorphism group of a simply connected, locally finite polyhedral complex X. Fix a quotient complex of groups $G(Y)$ induced by the action of Γ on X. By Theorem 2.1, we need to count only the number of isomorphism classes of coverings of faithful developable complex of groups by $G(Y)$.

From the definition of covering, as Y is a finite polyhedral complex, there exist only finitely many polyhedral complexes Z such that a covering (of any number of sheets) may be defined over a simplicial map $l : Y' \to Z'$ (since l must be surjective). Let N be the number of such lattices. Note that N is the number of all possible Z's and that for each fixed n, the number of possible Z's for which there is a covering $G(Y) \to H(Z)$ of index n is bounded above by N.

Now let us count the number of isomorphism classes of n-sheeted coverings of complexes of groups $\lambda : G(Y) \to H(Z) = (H_\tau, \psi_{a'}, g_{a',b'})$ over morphisms $l : Y \to Z$, where Z is fixed. Note that some of the complexes of groups defined on Z might result in isomorphic coverings; thus, our counting is valid only as an upper bound.

For $\sigma \in V(Y')$, let $c_\sigma = |G_\sigma|$, and for $\tau \in V(Z')$, let

$$c_\tau = \left(\sum_{\sigma \in f^{-1}(\tau)} c_\sigma^{-1} \right)^{-1}.$$

By the definition of an n-sheeted covering, the cardinality $|H_\tau|$ is equal to nc_τ. Let $c_0 = |V(Y')| \geq |V(Z')|$ and $c_1 = |E(Y')| \geq |E(Z')|$.

Let us first count the number of possible complexes of groups $H(Z)$. There are at most $\prod_{\tau \in V(Z')} (|H_\tau|)^{g(|H_\tau|)}$ isomorphism classes of groups H_τ. There are at most $\prod_{b \in E(Z')} (|H_{t(b)}|)^{\mu(|H_{i(b)}|)+1}$ monomorphisms $\psi_b : H_{i(b)} \to H_{t(b)}$ determined by the images of generators of $H_{i(b)}$, and at most $\prod_{a \in E(Z')} (|H_{t(a)}|)^{c_1}$ twisting elements $g_{a',b'}$. Now for a given complex of groups $H(Z)$, we count the number of possible coverings determined by local maps λ_σ and elements

$\lambda(a)$. There are at most $\prod_{\sigma\in V(Y')}(|H_{l(\sigma)}|)^{\mu(c_\sigma)+1}$ monomorphisms $\lambda_\sigma : G_\sigma \to H_{l(\sigma)}$ and at most $\prod_{a\in E(Y')}|H_{t(l(a))}|$ choices for the $\lambda(a)$.

Let $M = \max_{\sigma\in V(Y')}\max\{c_\sigma, c_{l(\sigma)}\}$ and $\mu = \mu(Mn)$. The number $u_\Gamma(n)$ is at most the product of the number of isomorphism classes of groups H_τ, the number of monomorphisms ψ_b, the number of twisting elements $g_{a',b'}$, the number of local maps λ_σ, and the number of elements $\lambda(a)$. Combining all the estimates above, we get the following upper bound for $u_\Gamma(n)$:

$$\begin{aligned} u_\Gamma(n) \quad &\leq \quad N\Pi_{\tau\in VZ'}(c_\tau n)^{g(c_\tau n)}\Pi_{b\in EZ'}(c_{t(b)}n)^{\mu(c_{i(b)}n)+1+c_1} \\ &\qquad \Pi_{\sigma\in VY'}(c_{l(\sigma)}n)^{\mu(c_\sigma)+1}\Pi_{a\in EY'}c_{t(l(a))}n \\ &\leq \quad N(Mn)^{c_0 g(Mn)+c_1(\mu(Mn)+c_1+1)+c_0(\mu(M)+1)+c_1} \\ &\leq \quad N(Mn)^{C_1\mu(Mn)^2} \leq (C_0 n)^{C_1'(\log n)^2} \end{aligned}$$

where $C_0 = MN$, $C_1 = c_0(2/27+1/2+75+16+1+1)+c_1(1+c_1+2)$, and $C_1' = 4C_1(\mu(M)+1)^2$. Note that the last inequality comes from the fact that $\mu(Mn) \leq \mu(M)+\mu(n) \leq (\mu(M)+1)2\log n$. (Here, log denotes the natural log.) This completes the proof of Theorem 1.1. □

Remark 3.1. (1.) The leading term comes from the number of isomorphism classes of local groups H_τ. More careful counting of other morphisms or twisting elements does not change the asymptotics of the upper bound.

(2.) We do not insist on the complex of groups $H(Z)$ being faithful or developable, which indicates that a better general upper bound might be obtained. However, the faithfulness condition seems to translate into a hard question in finite group theory. For example, in David M. Goldschmidt's deep result [10], the finite number of amalgams comes from faithfulness of certain graphs of groups. Indeed, the Goldschmidt-Sims Conjecture (the analogue of [10] for more general amalgams) has been open for several decades.

(3.) Here is an alternative proof of Theorem 1.1, using the upper bound in [14].

Let X and Γ be as in Theorem 1.1. Let $X^{(1)}$ be the 1-skeleton of X, $N = \pi_1(X^{(1)}, \cdot)$ its (free) fundamental group, and $T = \widetilde{X^{(1)}}$ its universal covering tree. There is a short exact sequence

$$1 \to N \to H = N\,\mathrm{Aut}(X) \to \mathrm{Aut}(X) \to 1,$$

and an action of H on the tree T, giving rise to an embedding $H < \mathrm{Aut}(T)$. The group $\Gamma N < H < \mathrm{Aut}(T)$ is a uniform lattice in H; thus, the number of overlattices of $\Gamma < \mathrm{Aut}(X)$ is bounded by the number of overlattices of ΓN in $\mathrm{Aut}(T)$.

Note that to prove the upper bound for tree overlattices in [14], standard Bass-Serre theory does not suffice, and similar results to those given in subsection 2.3 had to be proved. Therefore, given the results in [15], this alternative proof roughly amounts to our main proof.

3.2. Lower bound for right-angled buildings

We now prove Theorem 3.2 below, a special case of which was stated as Theorem 1.2 in the introduction. This theorem gives a lower bound on the number of overlattices of a particular lattice for certain right-angled buildings. See subsection 2.4 for definitions.

Theorem 3.2. *Let X be a regular right-angled building of type (W, S) and parameters $\{q_s\}_{s\in S}$. Assume that for some $t, t' \in S$, with $t \neq t'$,*

(1) *$q_t = q_{t'} = 2p$ where p is prime; and*
(2) *$m_{tt'} = \infty$.*

Then there is a cocompact lattice Γ in $\mathrm{Aut}(X)$, acting without inversions, such that for $n = p^k$ and $k \geq 3$,

$$u_\Gamma(n) \geq n^{\frac{1}{2}(k-3)}.$$

Let X be as in Theorem 3.2. Let T_{2p} be the $2p$-regular tree. In [14], Lim constructed many non-isomorphic coverings of faithful graphs of groups with universal cover T_{2p}, as sketched in Figure 1.

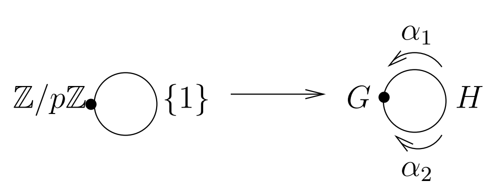

Figure 1. Coverings of faithful graphs of groups with universal cover T_{2p}

Let $\Gamma_{2p} < \mathrm{Aut}(T_{2p})$ be the cocompact lattice which is the fundamental group of the left-hand graph of groups in Figure 1. The lower bound obtained from these constructions is $u_{\Gamma_{2p}}(n) \geq n^{\frac{1}{2}(k-3)}$, for $n = p^k$ and $k \geq 3$.

To obtain the same lower bound for overlattices of a certain cocompact lattice Γ in $\mathrm{Aut}(X)$, we first take the "double cover" of the graphs of groups in Figure 1, as shown in Figure 2.

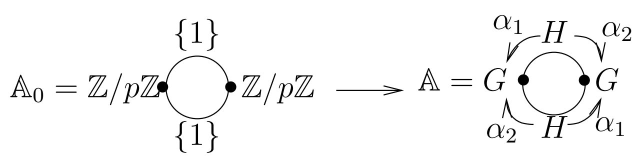

FIGURE 2. "Double cover" of graphs of groups in Figure 1

We now carry out a special case of the Functor Theorem of [21], for the graphs of groups $\mathbb{A}_0$ and $\mathbb{A}$ in Figure 2. The idea is to "fatten up" these graphs of groups to obtain complexes of groups $G_0(Y)$ and $G(Y)$, respectively, so that the fundamental group Γ of $G_0(Y)$ is a cocompact lattice in $\mathrm{Aut}(X)$ and the fundamental group of $G(Y)$ is an overlattice of Γ.

Let A be the graph underlying both of the graphs of groups in Figure 2. Let K be the cone on the barycentric subdivision of the nerve for (W, S), as described in subsection 2.4. Note that since $m_{tt'} = \infty$, the t- and t'-mirrors of K are disjoint. Let K^1 and K^2 be two copies of K. Glue together, preserving types, the t-mirrors of K^1 and K^2, and similarly with the t'-mirrors. Denote by Y the resulting polyhedral complex. The vertices of Y have well-defined types $T \in \mathcal{S}$, induced by the types of the vertices of K^1 and K^2. The edges of Y are then naturally oriented, so that an edge a of Y joining vertices of types $T, T' \in \mathcal{S}$ has $i(a)$ of type T and $t(a)$ of type T' if and only if $T \subsetneq T'$. Composable edges may then also be defined, and thus we may define a complex of groups over Y (without first taking the barycentric subdivision).

Choose an identification of the two vertices of the graph A with the vertices of Y of types $\{t\}$ and $\{t'\}$ and of the two edges of A with the two vertices of Y of type $\emptyset$. An example is sketched on the left of Figure 3, which shows the complex Y, and the types of its vertices, for K the barycentric subdivision of a regular right-angled hyperbolic hexagon with K_t and $K_{t'}$ on opposite sides of the hexagon.

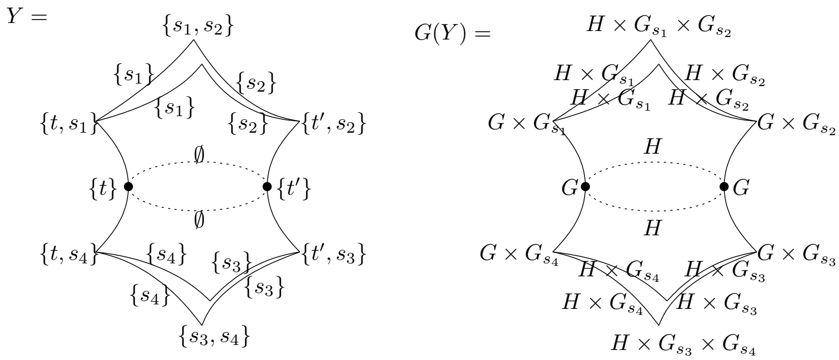

FIGURE 3. The space Y and the complex of groups $G(Y)$

We now explain how $\mathbb{A}$ induces a complex of groups $G(Y)$ over Y. The construction for $\mathbb{A}_0$ is similar. First, fix the local groups G and H induced by the identification of the vertices and edges of A with certain vertices of Y. Each of the monomorphisms $\alpha_i : H \to G$ in $\mathbb{A}$ then induces a monomorphism ψ_a along an edge a of Y with $i(a)$ of type $\emptyset$ and $t(a)$ of type either $\{t\}$ or $\{t'\}$.

To assign the remaining local groups and monomorphisms in $G(Y)$, for each $s \in S$, let G_s be a group of order q_s. Let $T \in \mathcal{S}$. If T contains neither t nor t', then the local group at the two vertices of Y of type T is

$$H \times \prod_{s \in T} G_s.$$

The monomorphisms between such local groups are natural inclusions. Now suppose T contains one of t and t'. (Since $m_{tt'} = \infty$ and W_T is finite, T cannot contain both t and t'.) Without loss of generality, suppose T contains t. Then the unique vertex of type T in Y is contained in the glued t-mirror, and we assign the local group at this vertex to be

$$G \times \prod_{s \in T, s \neq t} G_s.$$

The monomorphism from G (the local group at the vertex of Y of type $\{t\}$) to this local group is inclusion onto the first factor. For each $T' \subset T$ with $t \in T'$, the monomorphism

$$G \times \prod_{s \in T', s \neq t} G_s \to G \times \prod_{s \in T, s \neq t} G_s$$

is the natural inclusion. For each $T' \subset T$ with $t \notin T'$, the monomorphism

$$H \times \prod_{s \in T'} G_s \to G \times \prod_{s \in T, s \neq t} G_s$$

is a monomorphism $\alpha_i : H \to G$ from the graph of groups $\mathbb{A}$ on the first factor, with $i = 1, 2$ chosen so that all triangles of monomorphisms commute, and natural inclusions on the other factors. Put all $g_{a,b} = 1$, and we have a complex of groups $G(Y)$. See the right of Figure 3 for an example. Let $G_0(Y)$ be the complex of groups induced in the same way by $\mathbb{A}_0$.

We now show that $G_0(Y)$ and $G(Y)$ have nonpositive curvature and are thus developable, and that they have universal cover the regular right-angled building X. For this, we claim first that the link of each vertex of $G_0(Y)$ of type T in its local development is the join of $|T|$ sets of points of cardinalities, respectively q_s, $s \in T$. The only vertices where this requires some care are those of type T where T contains either t or t'; here there are two collections of cosets of the trivial group $\{1\}$ in $\mathbb{Z}/p\mathbb{Z}$ (or of H in G for $G(Y)$), each of cardinality p, and since $q_t = q_{t'} = 2p$, the claim follows. For example, in Figure 3, the local development at the vertex of type $T = \{t, s_1\}$ is the (barycentric subdivision of) the complete bipartite graph $K_{q_{s_1}, q_t}$; the cosets in $G \times G_{s_1}$ of G contribute q_{s_1} vertices, and the cosets in $G \times G_{s_1}$ of the two copies of $H \times G_{s_1}$ contribute $q_t = 2p$ vertices.

With the standard piecewise Euclidean metric on K, which induces a metric on Y, it follows by Gromov's Link Condition that $G_0(Y)$ and $G(Y)$ have nonpositive curvature and are thus developable. The universal cover of $G_0(Y)$ and of $G(Y)$ is a building of type (W, S) (see [8, Section 3.3]), and by construction it is regular. Theorem 2.5 then implies that the universal cover of $G_0(Y)$ and of $G(Y)$ is the unique regular right-angled building X of type (W, S) and parameters $\{q_s\}$. Hence, $G_0(Y)$ and $G(Y)$ are developable complexes of groups with universal cover X.

By construction, every covering, as in Figure 2, induces a covering of complexes of groups $G_0(Y) \to G(Y)$. Recall that the graphs of groups in [14] are faithful because there is no nontrivial subgroup of H whose images in G under α_1 and α_2 are the same. This condition implies that there is no nontrivial group N_T satisfying the conditions in Proposition 2.2; thus, each $G(Y)$ is faithful. Moreover, in Lim's construction, distinct coverings of the form in Figure 1 are non-isomorphic because the vertex and edge groups G and H are non-isomorphic; thus, they induce non-isomorphic coverings $G_0(Y) \to G(Y)$. By Theorem 2.1, this completes the proof. $\square$

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(Lim) Department of Mathematical Sciences; Seoul National University; Seoul, Korea 151-747
E-mail address: slim@snu.ac.kr

(Thomas) Mathematical Institute; University of Oxford; 24-29 St Giles'; Oxford OX1 3LB United Kingdom
E-mail address: anne.thomas@maths.ox.ac.uk