Representations of the orthosymplectic Yangian

Alexander Molev

University of Sydney

Plan

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- Yangian for $\mathfrak{gl}_N$ and its representations

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- Yangian for $\mathfrak{osp}_{N|2m}$ in the RTT presentation

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- Explicit construction of representations of $\mathrm{Y}(\mathfrak{osp}_{1|2})$

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- Explicit construction of representations of $\mathrm{Y}(\mathfrak{osp}_{1|2})$
- Classification theorems for $\mathfrak{osp}_{1|2m}$ and $\mathfrak{osp}_{2|2m}$

Yangian for $\mathfrak{gl}_N$

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The Yangian for $\mathfrak{gl}_N$ is the associative algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $i,j=1,\dots,N$, and the defining relations

$$\big[t_{ij}^{(r+1)},t_{kl}^{(s)}\big]-\big[t_{ij}^{(r)},t_{kl}^{(s+1)}\big]=t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)},$$

where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$.

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where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$.

This algebra is denoted by $\mathrm{Y}(\mathfrak{gl}_N)$.

Introduce the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

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The defining relations take the form

$$(u - v)\left[t_{ij}(u), t_{kl}(v)\right] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u) :$$

equate the coefficients of $u^{-r} v^{-s}$.

Introduce the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N,$$

where $e_{ij} \in \operatorname{End} \mathbb{C}^N$ are the standard matrix units.

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It satisfies the Yang–Baxter equation.

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$$T_1(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}(u) \quad \text{and} \quad T_2(u) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes t_{ij}(u).$$

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The defining relations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ can be written in the form of RTT-relation [Faddeev's school, 1980s]

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v).$$

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for some formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \dots, \qquad \lambda_i^{(r)} \in \mathbb{C}.$$

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The irreducible highest weight representation $L(\lambda(u))$ of $\mathrm{Y}(\mathfrak{gl}_N)$ with the highest weight $\lambda(u)$ is the quotient of the Verma module $M(\lambda(u))$ by the unique maximal proper submodule.

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The irreducible highest weight representation $L(\lambda(u))$ of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is finite-dimensional if and only if

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)}, \qquad i = 1, \dots, N-1,$$

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[V. Tarasov 1985, V. Drinfeld 1988].

Super Yangians

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The vector e_i has the parity $\bar{\imath} \mod 2$ and

$$\bar{\imath} = \begin{cases} 1 & \text{for} \quad i = 1, \ldots, m, m', \ldots, 1', \\ 0 & \text{for} \quad i = m+1, \ldots, (m+1)', \end{cases}$$

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The endomorphism algebra $\operatorname{End} \mathbb{C}^{N|2m}$ is equipped with $\mathbb{Z}_2$-gradation, the parity of the matrix unit e_{ij} is $\bar{\imath} + \bar{\jmath} \mod 2$.

A standard basis of the Lie superalgebra $\mathfrak{gl}_{N|2m}$ is formed by elements E_{ij} of parity $\bar{\imath} + \bar{\jmath} \mod 2$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

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The orthosymplectic Lie superalgebra $\mathfrak{osp}_{N|2m}$ is the subalgebra of $\mathfrak{gl}_{N|2m}$ spanned by the elements

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where

$$\theta_i = \begin{cases} 1 & \text{for} \quad i = 1,\dots,N+m, \\ -1 & \text{for} \quad i = N+m+1,\dots,N+2m. \end{cases}$$

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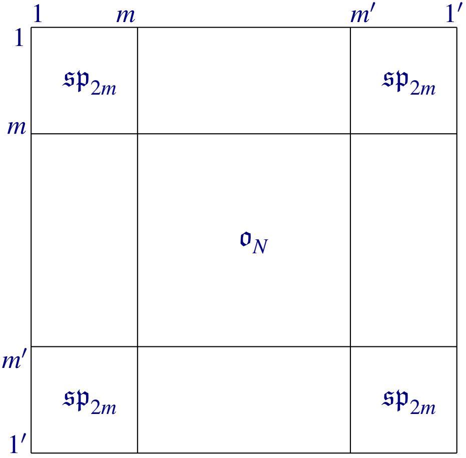

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The R-matrix associated with $\mathfrak{osp}_{N|2m}$ is the rational function in u given by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa}, \qquad \kappa = \frac{N}{2} - m - 1.$$

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[A. B. Zamolodchikov and Al. B. Zamolodchikov, 1979]

The extended Yangian $\mathrm{X}(\mathfrak{osp}_{N|2m})$ as a $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar{\imath} + \bar{\jmath} \mod 2$, where $1 \leqslant i, j \leqslant N + 2m$ and $r = 1, 2, \dots$, satisfying the following defining relations.

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Introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{X}(\mathfrak{osp}_{N|2m})[[u^{-1}]]$$

and combine them into the matrix $T(u) = [t_{ij}(u)]$.

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[D. Arnaudon, J. Avan, N. Crampé, L. Frappat, E. Ragoucy, '03]

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Theorem. The Verma module $M(\lambda(u))$ is nonzero if and only if

$$\lambda_i(u)\,\lambda_{i'}\Big(u - \frac{N}{2} - (-1)^{\bar{\imath}}(m-i) + 1\Big) = \lambda_{i+1}(u)\,\lambda_{(i+1)'}\Big(u - \frac{N}{2} - (-1)^{\bar{\imath}}(m-i) + 1\Big)$$

for $1 \leqslant i < m + N/2$.

Hence we can re-define the highest weight by

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Theorem. Every finite-dimensional irreducible representation of the Yangian $\mathrm{X}(\mathfrak{osp}_{N|2m})$ is isomorphic to a unique irreducible highest weight representation $L(\lambda(u))$.

Open problem. Find necessary and sufficient conditions on $\lambda(u)$ for the representation $L(\lambda(u))$ to be finite-dimensional.

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M. Nazarov 1991, R. Zhang 1996, A. M. 2022.

Solution for $\mathfrak{osp}_{1|2}$

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The solution relies on an explicit construction

of the modules $L(\alpha)$.

Small Verma modules

Small Verma modules

Let K be the submodule of $M(\lambda(u))$ generated by all vectors

$$t_{21}^{(r)}\,\xi \quad \text{for} \quad r \geqslant 2 \quad \text{and} \quad \big(t_{31}^{(r)} + (\alpha - 1/2)\,t_{31}^{(r-1)}\big)\,\xi \quad \text{for} \quad r \geqslant 3,$$

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Proposition. The elementary module $L(\alpha)$ is a quotient of the small Verma module $M(\alpha)$.

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For any $r, s \in \mathbb{Z}_+$ introduce vectors in $M(\alpha)$ by

$$\xi_{rs} = T_{21}(-\alpha-r+3/2)\dots T_{21}(-\alpha-1/2)\,T_{21}(-\alpha+1/2)$$

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Proposition. For any $\alpha \in \mathbb{C}$ the vectors ξ_{rs} with $0 \leqslant r \leqslant s$ form a basis of $M(\alpha)$.

Basis diagram of $M(\alpha)$

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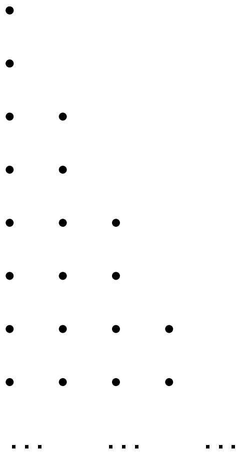

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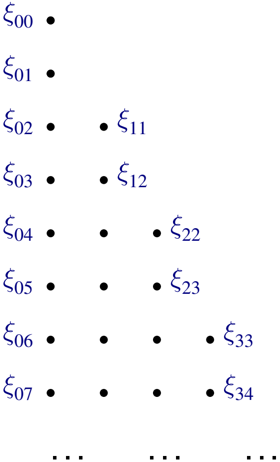

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- The $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $L(\alpha)$ is finite-dimensional if and only if $-\alpha = k \in \mathbb{Z}_+$. In this case,

$$\dim L(-k) = \binom{k+2}{2}.$$

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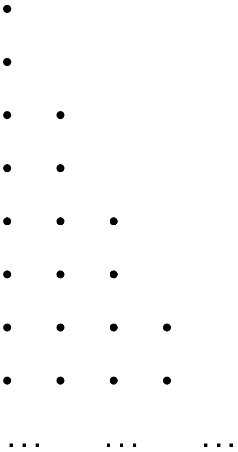

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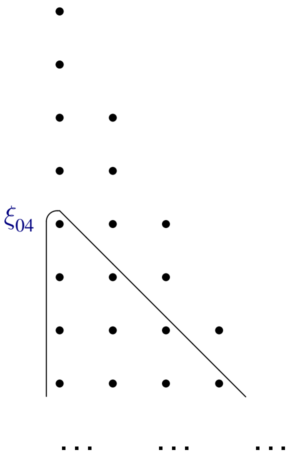

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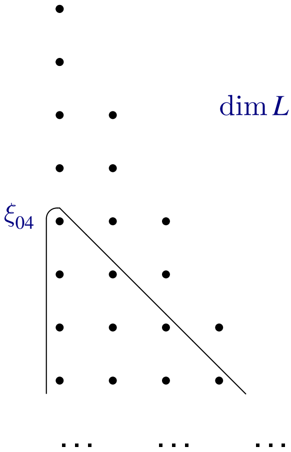

$$\dim L(-3) = \binom{5}{2} = 7 + 3 = 10.$$

For any given $\mu \in \mathbb{C}$ denote by $V(\mu)$ the irreducible highest weight module over $\mathfrak{osp}_{1|2}$ generated by a nonzero vector ξ such that $F_{11}\xi = \mu\,\xi$ and $F_{12}\xi = 0$.

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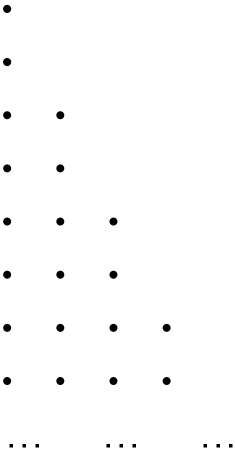

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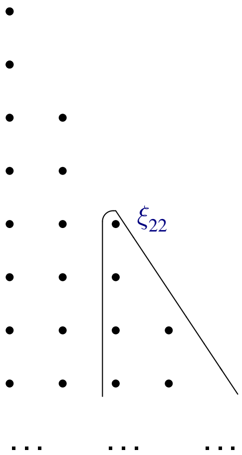

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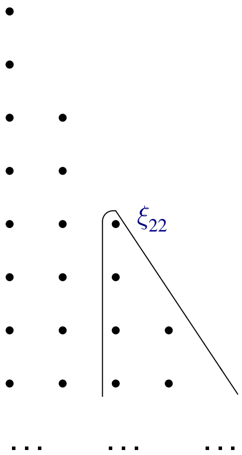

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Theorem. The representation $L(\lambda(u))$ is finite-dimensional if and only if

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$$\frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)} \qquad \text{for} \quad i = 2, \dots, m,$$

and

$$\frac{\lambda_{m+2}(u)}{\lambda_{m+1}(u)} = \frac{P_{m+1}(u+2)}{P_{m+1}(u)}. \qquad \text{Alg. Rep. Th., online.}$$