The University of Sydney

How to multiply pictures, and why

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What is this subject about?

In 1679, the German mathematician Gottfried von Leibnitz stated that:

"We need another kind of analysis which deals directly with position, as algebra deals with magnitude"

He called this new analysis "*analysis situs*"; it is now known as topology.

Gauss: showed that the current induced in a wire by the magnetic field created by the current in another wire is proportional to the "winding number" of the 2 wires, i.e. the number of times they are intertwined. Winding number is a topological concept.

Knot theory, a branch of topology, was created by 4 physicists: Maxwell, Lord Kelvin (William Thomson), Helmholtz, and Peter Tait in about 1850. They had various motivations, including the explanation of all physical interactions.

The problem which Lord Kelvin and Tait thought in 1867 would be straightforward to solve, viz. to give a complete classification of all knots, remains unsolved, and important, to this day.

Today, "algebra", is no longer concerned with magnitude in the sense that Leibnitz used the word. It is concerned with mathematical structure, i.e., the structure of abstract systems.

In this talk, we'll see an emerging subject which is an instance of Leibnitz's 'new analysis'-a mixture of geometry and algebra.

Algebra

My aim is: **to do algebra with diagrams**, just like we do with numbers, matrices, etc.

To begin, let's see in a very simple minded, practical, way what the term "algebra" means

"Algebra" is simply an environment where we may perform manipulations which have familiar properties;

The ingredients we shall need are firstly, a set $\mathcal{A}$ ("an algebra") together with a base field F of coefficients (the scalars), which you may take to be the real or complex numbers.

As usual in mathematics, what the symbols represent doesn't matter. The important thing is what you can do with them.

We need to be able to do three things: multiply, add, and "multiply by a scalar". The last operation tells us how $\mathcal{A}$ and F interact.

Why? Because in this environment we can use a combination of arguments, some formal and algebraic, and some coming from geometric intuition.

EXAMPLE: take $\mathcal{A}$ to be the set of points in the plane, $\mathbb{R}^2$ and the base field $F = \mathbb{R}$, i.e. $\mathcal{A} = \{(x, y) \mid x, y \in \mathbb{R}\}$.

Addition and multiplication are defined coordinate-wise: $(x, y) + (x', y') = (x + x', y + y')$ and $(x, y).(x', y') = (xx', yy')$. Scalar multiplication is defined in the usual (obvious) way: for $\alpha \in \mathbb{R}$, $\alpha(x, y) = (\alpha x, \alpha y)$.

These operations satisfy the usual rules which apply to operations with numbers, with which you are familiar; for example, we have the following rule for multiplying linear combinations:

For $a, b, a', b' \in \mathcal{A}$ and $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$,

$$(\alpha a + \beta b)(\alpha' a' + \beta' b') = \alpha\alpha' aa' + \alpha\beta' ab' + \beta\alpha' ba' + \beta\beta' bb'.$$

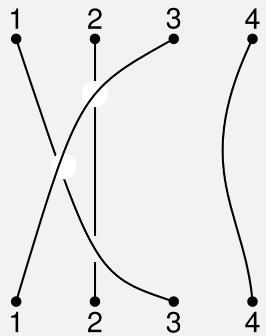
Notice that in the example, $ab = ba$ for all $a, b \in \mathcal{A}$. But this is not true in general, and will not be true in most of the algebras coming up soon. But the equation above is always true. That is why I have written it in the form above.

Diagrams

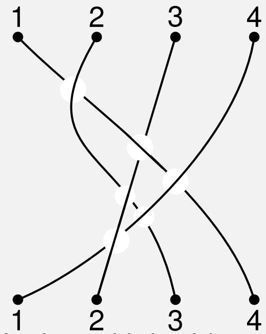
We are going to explore several algebras $\mathcal{A}$ whose elements include diagrams (pictures)

We therefore start by looking at some diagrams, and how to multiply them.

The first type of diagram we shall consider is a "braid"; here is a picture of a 4-string braid, which we'll call b_1:



Here is a second 4-string braid, b_2:



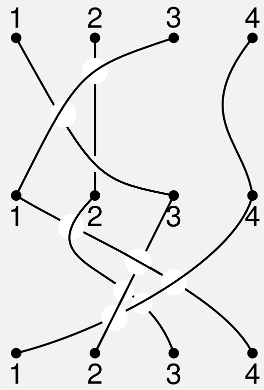
We want to do algebra with braids, so we need to see how to multiply 2 braids, add them, and how to multiply a braid by a scalar.

First, let's deal with the most important operation: multiplication. Note that here we fix the number of strings (in our illustration, 4).

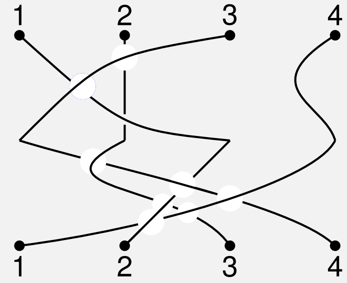
Multiplying braids

To multiply two *n*-string braids, we simply concatenate them.

For example, take the 2 braids we have seen, and join them thus:



Then delete the intermediate nodes, to obtain a new braid:

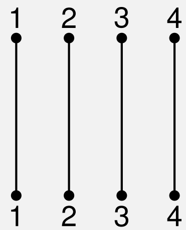


The above braid is the product $b_1 b_2$ of the braids b_1 and b_2. It's easy to see that it is generally false that $b_1 b_2 = b_2 b_1$.

But the multiplication also clearly satisfies the associative law: $(b_1 b_2) b_3 = b_1 (b_2 b_3)$.

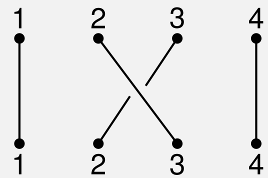
Elementary braids, equations.

The easiest braid of all is the "trivial braid" e:



It is obvious that composing any braid with the trivial one leaves that braid unchanged; i.e. for any braid b, $eb = be = b$.

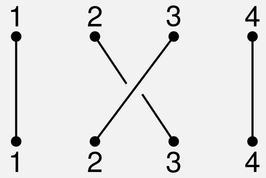
The picture below shows the "elementary braid" σ_2.



In general, for *n*-string braids, we have $\sigma_1, \ldots, \sigma_{n-1}$.

Arbitrary braids from elementary ones

The picture below shows the braid σ_2^{-1}.



Similarly, we have σ_i^{-1} for any i

The next assertion is easy to see, but important.

Proposition

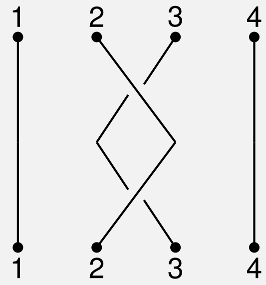
Any *braid b can be obtained by composing a sequence of elementary braids* i.e. $b = \sigma_{i_1}^{\pm 1}\sigma_{i_2}^{\pm 1} \dots \sigma_{i_p}^{\pm 1}$

Relations among the elementary braids

We shall quickly see that the representation of a braid as a composition of a sequence of elementary ones is far from unique.

But remarkably, we'll have precise (but not complete) control over this non-uniqueness.

Let's start by composing σ_2 with σ_2^{-1}:



As you can see by pulling the second string across, over the third, $\sigma_2\sigma_2^{-1} = e$, and similarly $\sigma_2^{-1}\sigma_2 = e$.

These are the first examples of equations in our algebra.

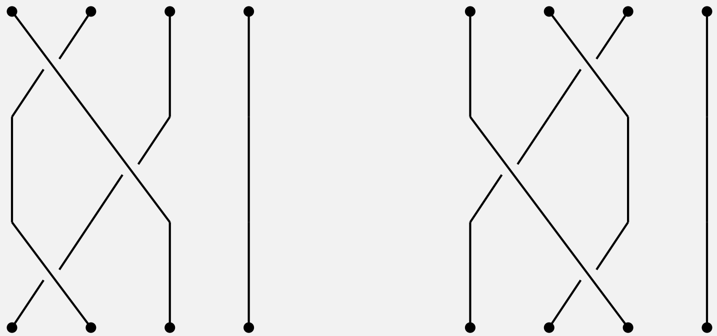
Here is another: $\sigma_1\sigma_3 = \sigma_3\sigma_1$.

This is obvious, because the strings 'moved' by σ_1 and σ_3 have nothing to do with each other.

In general, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| \geq 2$.

The classical braid relations

Let us compare $\sigma_1\sigma_2\sigma_1$ with $\sigma_2\sigma_1\sigma_2$:



$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$

They are equal! That is, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

We've now seen the two fundamental relations which the elementary braids σ_i satisfy, viz:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2,$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i.$$

Of course we also have $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = e$.

The relations are easy to prove, but the theorem I shall now state is not:

Artin's theorem

Theorem

Given any representation of a braid as a product of elementary braids and their inverses, $b = \sigma_{i_1}^{\pm 1} \ldots \sigma_{i_p}^{\pm 1}$, any other such representation of b can be obtained from the given one by a sequence of formal manipulations, using only the two given fundamental relations.

Example: $\sigma_2^{-1} \sigma_4 \sigma_1 \sigma_2 \sigma_4^{-1} \sigma_1^2 = \sigma_1 \sigma_2 \sigma_1$.

Another way of stating this:the relations we have seen are essentially "all relations" among braids.

Artin first proved this in 1926, then again in 1947 and 1956; all proofs require ideas from several branches of mathematics.

Addition and scalar multiplication

So far, we have seen how to multiply braids. But I promised you 3 operations; so what about addition and scalar multiplication?

Our algebra $\mathcal{A}$ will actually have elements which are (formal) linear combinations of braids: $a = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_p b_p$,

where the $\alpha_i \in F$ are scalars, and the $b_i \in B_n$ are all n-string braids.

The definition of addition and scalar multiplication is now obvious (although you may feel cheated).

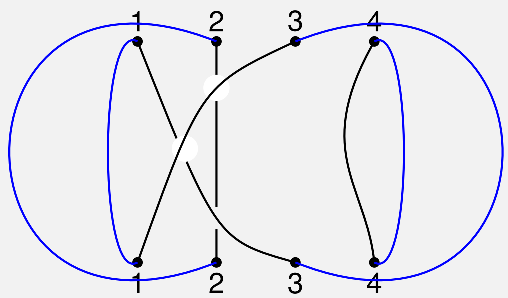
But the definition of multiplication now must be extended to linear combinations. It can only be done in one way if we are to satisfy the axioms discussed earlier:
$(\sum_i \alpha_i b_i)(\sum_j \alpha'_j b'_j) = \sum_{i,j} \alpha_i \alpha'_j b_i b'_j$.

An application: from braids to links and knots

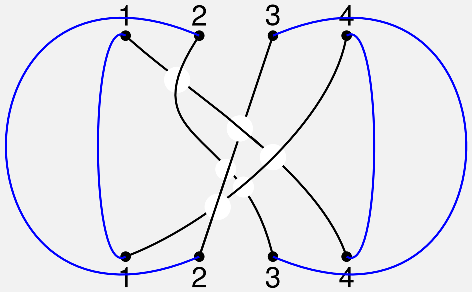
Why do this? There are many applications; here is one which leads to the celebrated "Jones invariant" of knots.

Any braid can be made into a link–i.e. an embedding of a finite set of circles in $\mathbb{R}^3$, which are possibly linked and/or knotted. If there is just one circle, we call the link a knot.

Start with a braid: (we'll take b_1 from earlier) Then 'close' it by joining the corresponding top and bottom nodes:

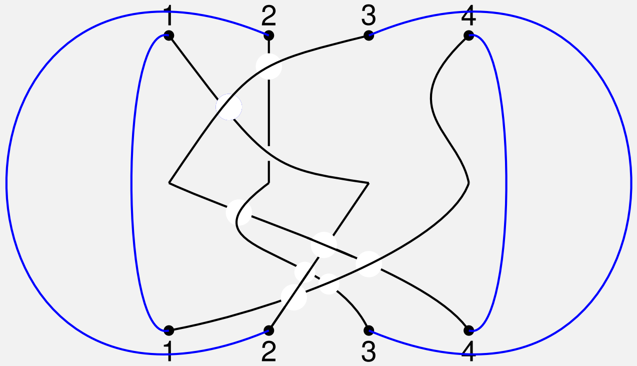


The first example above gives just 3 unlinked circles– not very interesting; let's try another example:



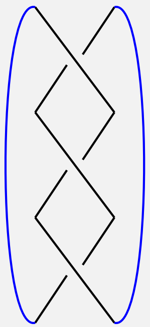
This yields two circles, which are unknotted, but linked.

However, if we take $b_1 b_2$, we get quite a complicated knot:



Here is another example, which may be familiar:

Start with $\sigma_1^3 \in B_2$, and close it:



This is the common trefoil knot.

Alexander's Theorem

The main point of this "closing of braids" construction is:

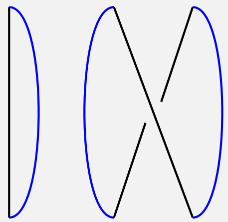
Theorem

Any (oriented) link can be obtained by closing an n-string braid, for some n.

If $\mathcal{B}$ denotes the set of **all** n-string braids, for $n = 1, 2, 3, \ldots,$, and $\mathcal{L}$ denotes the set of all links, we therefore have a **surjective** map $\mathcal{B} \to \mathcal{L}$

But the same link can come from many different braids!

For example, an unknotted circle comes from both the trivial 1-string braid, and from the 2-string braid σ_1:



We now attach a polynomial to each link (and therefore to each knot as a special case) as follows:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow[\text{braid}]{\text{close}} & \mathcal{L} \\ {\scriptstyle \sigma_i \mapsto \sigma_i}\downarrow & & \downarrow{\scriptstyle \text{invariant: } L \mapsto P_L(\ell, m)} \\ \mathcal{A}_0 & \xrightarrow{\phi = \mu.\tau} & \mathbb{C}[\ell^{\pm 1}, m^{\pm 1}], \end{array}$$

To define the invariant polynomial P_L:

- start with a link L, take any braid b whose closure is L
- regard b as an element of the algebra $\mathcal{A}_0$ obtained from the braid algebra $\mathcal{A}$ by adding relations.
- evaluate the “trace function” τ at b; τ arises from the algebraic structure.
- multiply $\tau(b)$ by $\mu(b)$ to ensure that the result does not depend on the choice of b

Note that if $\mathcal{A}(n)$ is the algebra of n-string braids, then $\mathcal{A}(1) \subset \mathcal{A}(2) \subset \mathcal{A}(3) \subset \ldots$. I have skirted around the possible dependence on the number of strings n.

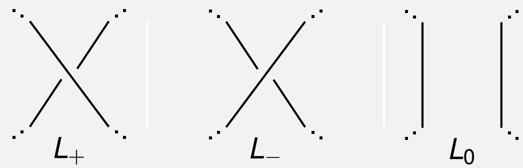
The algebra $\mathcal{A}_0$ is obtained by imposing some extra relations on $\mathcal{A}$.

The extra relations are of the form $\sigma_i^2 + x\sigma_i + y = 0$ for suitable non-zero elements x, y of F; this says merely that the σ_i satisfy a quadratic equation.

This quadratic relation makes $\mathcal{A}_0$ into a “Hecke algebra”, which is a famous class of algebras, about which much is known.

Geometrically, this quadratic relation means that the link invariant $P_L = P_L(\ell, m)$ satisfies a “skein relation”; that is:

If L_+, L_- and L_0 are links which are the same everywhere except at one crossing, where they look like:



The invariants of the 3 links are then related by:

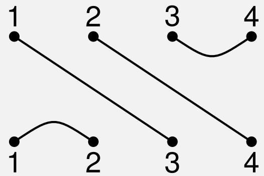
$$\ell P_{L_+} + \ell^{-1} P_{L_-} + m P_{L_0} = 0.$$

This “skein relation” provides an easy way of calculation for P_L by unravelling the knot, crossing by crossing.

The skein relation is equivalent to the the quadratic relation we have seen for the braids σ_i.

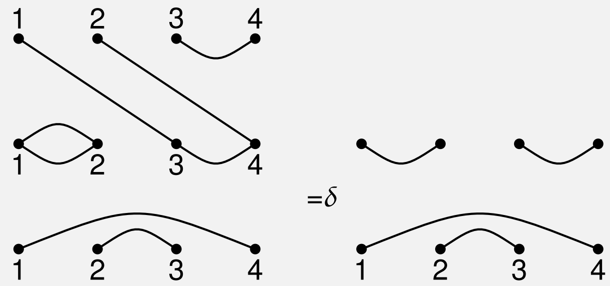
Other diagram algebras

We may impose further relations on the elements of the Hecke algebra, obtaining a “Temperley-Lieb algebra”, which also has elements which are diagrams, but are not braids. Here is such a TL diagram:



Temperley-Lieb algebras were invented by physicists to study phase changes in matter.

There is a calculus of “planar diagrams”, similar to that which we have seen for braids.



Where $\delta \in F$ is a parameter.

There are many other types of diagram algebras; we have diagrams on cylinders, or other surfaces. There are the mysterious “Feynman diagrams”, partition diagrams and many others.

Most of the algebras I have mentioned are “cellular”.

This means they come in families, which are parametrised by variables which range over some geometric space X; (above: $p = \delta \in F$). The structure of the algebra corresponding to a point $p \in X$ mostly varies continuously with p, but sometimes doesn’t. Thus there are “singular points”.

It is these singular points which are both mathematically and physically interesting. This is where “modular representation theory” and phase changes in physics meet; these subjects have never had any previous contact at all.

From vortices in the ether to modern mathematics

About 150 years ago, a group of celebrated physicists, including Lord Kelvin and Peter Tait set themselves the following program:

1. Classify all knots, and order them by complexity.
2. Determine which of them occurs as “vortex knots” in chemicals.
3. Explain the spectrum of a chemical in terms of knots.
4. Explain physical laws which matter follows in these terms.

This program itself has not gone according to plan; Step 1 is still not complete, but has led to many very rich mathematical veins.

The “algebra of diagrams” is an emerging subject in mathematics, which combines ideas from all 3 traditional areas: geometry, algebra and analysis.

Among the subjects which are impacted by this theory are:

- String theory in physics.
- Classification of ‘manifolds’.
- Theory of quantum groups and Hecke algebras.
- Quantum computing (cf. Michael Freedman at Microsoft)
- Enumerative geometry (counting intersections with multiplicity).
- Configuration spaces.
- Statistical mechanics.

HAPPY NEW YEAR!

I hope you have seen a glimpse of how ideas from diverse areas can come together to create something of value, and which is fun to play with.

I wish you all a bright future of playing with ideas.

HAPPY NEW YEAR!