ON LOCALLY CONFORMALLY FLAT GRADIENT SHRINKING RICCI SOLITONS

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ABSTRACT. In this paper, we first apply an integral identity on Ricci solitons to prove that closed locally conformally flat gradient Ricci solitons are of constant sectional curvature. We then generalize this integral identity to complete noncompact gradient shrinking Ricci solitons, under the conditions that the Ricci curvature is bounded from below and the Riemannian curvature tensor has at most exponential growth. As a consequence of this identity, we classify complete locally conformally flat gradient shrinking Ricci solitons with Ricci curvature bounded from below.

1. INTRODUCTION AND MAIN THEOREM

The study of Ricci soliton has been an important part in the study of the Ricci flow. It usually serves as a (dilation) limit of solutions to the Ricci flow. Though being a Ricci soliton is a purely static condition, it is usually convenient to view it as a solution to the Ricci flow.

Definition 1.1. A complete n-dimensional Riemannian manifold (M, g) is called a Ricci soliton if

$$2R_{ij} + \nabla_i V_j + \nabla_j V_i = 2\rho g_{ij} , \tag{1}$$

for some vector field V and constant ρ, where R_{ij} is the Ricci curvature tensor. Moreover, if V is a gradient vector field of a function f on M, then we have a gradient Ricci soliton, satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij} . \tag{2}$$

We say that (M, g) is expanding, steady or shrinking if ρ is $< 0, = 0$ or > 0, respectively. Notice that when f is a constant function, we have the Einstein equation.

The classification of gradient Ricci solitons has been a very interesting problem. For closed expanding and steady gradient Ricci solitons, it is well-known that they must be Einstein (see [Per02] or [CCG+07], Proposition 1.13). The shrinking case is a little bit more complicated. When $n = 2, 3$, it is known that closed shrinking Ricci solitons are Einstein (see [Ham88] and [Ive93]). If $n > 3$, there exist shrinking Ricci solitons which

are not Einstein (See N. Koiso [Koi90], H.-D. Cao [Cao96], M. Feldman, T. Ilmanen and D. Knopf [FIK03]). The recent work of C. Böhm and B. Wilking ([BW08]) implies that gradient Ricci solitons with positive curvature operator must be of constant curvature.

The classification of complete noncompact gradient Ricci solitons has been studied by many authors very recently under various conditions, for example, see [Nab07], [NW07], [PW07] and [FMZ08].

In [Cao07], the first author proves several identities on closed gradient Ricci solitons, we now extend one of them to the case of complete noncompact gradient Ricci solitons. Since the proof of this identity requires integration by parts, we need to have some control on curvatures and the potential function f such that we can justify integration by parts. Our main theorem is the following:

Theorem 1. *Let (M,g) be a non-flat complete noncompact shrinking gradient Ricci soliton given by* (2)*, suppose that the Ricci curvature of (M,g) is bounded from below. Assume further that it satisfies*

$$(3) \qquad\qquad |R_{ijkl}|(x) \leqslant \exp(a(r(x)+1))$$

for some constant $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. Then the identity

$$(4) \qquad\qquad \int_M |\nabla \operatorname{Ric}|^2 e^{-f} d\mu = \int_M |\operatorname{div} \operatorname{Rm}|^2 e^{-f} d\mu$$

holds.

As an application of the above integral identity (4), we will consider gradient shrinking Ricci solitons (GSRS) which are locally conformally flat. Such classification has also been considered in [ELM06], [NW07] and [PW07]. We will then prove the following theorem for closed Ricci solitons.

Theorem 2. *Let (M,g) be a closed gradient shrinking Ricci soliton. Assume that (M,g) is locally conformally flat. Then (M,g) must be Einstein. Moreover, (M,g) is of constant sectional curvature.*

Remark 1.1. In this case, the problem has also been studied by M. Eminenti, G. La Nave and C. Mantegazza [ELM06] using a different method.

In the complete noncompact case, the identity (4) again yields a classification of locally conformally flat GSRSs with Ricci curvature bounded from below.

Theorem 3. *Let (M^n,g), $n \geqslant 3$, be a complete noncompact gradient shrinking soliton whose Ricci curvature is bounded from below. Assume that (M,g) is locally conformally flat. Then its universal cover is either $S^{n-1} \times \mathbb{R}$ or $\mathbb{R}^n$.*

Remark 1.2. In the case of locally conformally flat GSRSs, the condition of (3) is same as

$$|R_{ij}|(x) \leqslant \exp(a(r(x)+1)).$$

This condition is not needed in Theorem 3.

Remark 1.3. L. Ni and N. Wallach ([NW07]) proved a slightly different version of this theorem. P. Petersen and W. Wylie ([PW07]) also proved a similar result. In dimension 3, their results do not require that *M* to be locally conformally flat.

After we have finished the proof of Theorem 2, we saw that L. Ni and N. Wallach have just proved their theorem, from which we realize that our proof can be extended to the complete noncompact case. The approach here is different.

Remark 1.4. An early version of this paper by the first two authors appeared on arXiv July 2008 (arXiv:0807.0588v2). Most recently, L. Chen and W. Chen [CC08] informed us that they can improve our result based on a similar idea.

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 2 for closed gradient Ricci solitons. The proof makes use of an integral identity obtained by the first author [Cao07]. In Section 3, we will deal with the complete noncompact case. We first prove the above mentioned integral identity on complete gradient Ricci solitons (Theorem 1), then we will finish the proof of Theorem 3.

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2. COMPACT CASE

In this section, we will deal with the case of closed Ricci solitons and prove Theorem 2. We first recall the following lemma from [Cao07, Corollary 1]. Note that the proof only uses the fact that (M,g) is a closed gradient soliton.

Lemma 4. *Suppose that (M,g) is a closed gradient Ricci soliton, then we have*

$$(5) \qquad \int_M |\nabla \operatorname{Ric}|^2 e^{-f} d\mu = \int_M |\operatorname{div} \operatorname{Rm}|^2 e^{-f} d\mu .$$

As (M,g) is locally conformally flat, we have

$$R_{ijkl} = \frac{1}{n-2} \left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il} \right)$$

$$(6) \qquad\qquad - \frac{R}{(n-1)(n-2)} \left(g_{ik}g_{jl} - g_{il}g_{jk} \right) .$$

Moreover, the following identity holds (see [Eis49], Eq. (28.19)),

$$(7) \qquad \nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{2(n-1)} \left(\nabla_k R g_{ij} - \nabla_j R g_{ik} \right) .$$

On a closed Riemannian manifold (M,g), it follows from the second Bianchi identity that

$$(\operatorname{div} \operatorname{Rm})_{jkl} = \nabla_i R_{ijkl} = \nabla_i R_{klij} = -\nabla_k R_{ijli} - \nabla_l R_{ijik}$$

$$(8) \qquad\qquad = \nabla_k R_{jl} - \nabla_l R_{jk} .$$

As a consequence of (7) and (8), on a closed locally conformally flat gradient Ricci soliton, we arrive at

$$|\operatorname{div}\operatorname{Rm}|^2 = |\nabla_k R_{ij} - \nabla_j R_{ik}|^2 = \frac{1}{2(n-1)}|\nabla R|^2 .$$

The above identity and (5) now imply that

(9)
$$\int_M |\nabla \operatorname{Ric}|^2 e^{-f} d\mu = \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} d\mu .$$

Since we have

$$0 \leqslant \left| \nabla_k R_{ij} - \frac{1}{n} \nabla_k R g_{ij} \right|^2 = |\nabla \operatorname{Ric}|^2 - \frac{1}{n}|\nabla R|^2 ,$$

this implies

$$\frac{1}{n}|\nabla R|^2 \leqslant |\nabla \operatorname{Ric}|^2 .$$

Plugging this into (9), we obtain the following inequality

$$\frac{1}{n}\int_M |\nabla R|^2 e^{-f} d\mu \leqslant \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} d\mu .$$

But this is only possible if $n \leqslant 2$ or $\nabla R \equiv 0$. Since we have already assumed that $n \geqslant 3$, we must have $\nabla R = 0$. Hence that the scalar curvature R is constant. Take trace on both sides of (2), then we have

(10)
$$R + \Delta f = n\rho ,$$

take integration on both sides of (10), we have

$$(n\rho - R)\operatorname{Vol}(M) = \int_M \Delta f d\mu = 0 ,$$

here we use the fact that M is closed. Therefore $R = n\rho$, and then $\Delta f = 0$. So f must be a constant. From (2), we know that (M,g) is Einstein, i.e.,

$$R_{ij} = \frac{R}{n} g_{ij} .$$

Plugging this into (6) implies that

$$R_{ijkl} = \frac{R}{n(n-1)} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) ,$$

and R is a constant, so (M,g) is of constant curvature. This finishes the proof of Theorem 2.

3. COMPLETE NONCOMPACT CASE

In this section, we will first extend Lemma 4 to the case of complete noncompact gradient Ricci solitons (Theorem 1). We will prove that equality (4) is also true for a complete gradient shrinking soliton whose Ricci curvature is bounded from below. Then we will finish the proof of Theorem 3.

Before we prove Theorem 1, we will need the following lemmas.

Lemma 5 ([Cao07]). *Suppose (M, g) is a complete gradient Ricci soliton, then we have the identities*

$$\nabla_i(R_{ijkl}e^{-f}) = 0, \tag{11}$$

and

$$\nabla_i(R_{ik}e^{-f}) = 0 . \tag{12}$$

Lemma 6. *Assume the same hypothesis as in Theorem 1. For any $D > 0$, there exist constants $B > 0$ and $C > 0$ such that*

$$f(x) \geqslant min\{\frac{\rho}{2}r(x)^2, \frac{Dr(x)}{3\rho}\}, \tag{13}$$

and

$$f(x) \leqslant Cr^2(x) , \quad |\nabla f|(x) \leqslant Cr(x) \tag{14}$$

for $r(x) \geqslant B$. Here $r(x)$ is the distance function to some fixed point $O \in M$, and ρ is the same constant given by (2).

Proof. We consider a complete noncompact gradient shrinking soliton (M, g). In other words, over M, for some constant $\rho > 0$, we have

$$\nabla^2 f + \text{Ric} - \rho g = 0.$$

We also assume that $\text{Ric} \geqslant -Kg$ for some $K > 0$. In the following C's might stand for different positive constant, but they are uniformly chosen.

Fix a point $O \in M$. For any $x \in M$, set $dist_g(O, x) = r(x) = s_0$. Consider a minimal geodesic, $\gamma(s)$, from O to x with arc-length, s, as parameter. In the following, we'll identify the arc-length parameter with the point on M.

We only need to consider points whose distance to O is large enough. We will choose this constant B uniformly as in the statement of the lemma.

To begin with, we have $\nabla^2 f \leqslant Cg$ by soliton equation (2) and our assumption on the lower bound of Ricci curvature. Thus $f(x) \leqslant Cr(x)^2$ for x with $r(x)$ large enough. Moreover, from $R + |\nabla f|^2 - 2\rho f = 0$ [1] and the lower bound of scalar curvature R (from lower

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[1] This equation can be derived from the original Ricci soliton equation using second Bianchi identity, where we have normalized the right hand side to be 0.

bound of Ricci curvature), we have

$$|\nabla f|^2 \leqslant Cr^2.$$

It's only left to show lower bound of the growth for f as (13). Let $\{\gamma', E_1, \cdots, E_{n-1}\}$ be a parallel orthonormal basis along this geodesic. For $s_0 > 2$ and r_0 which is relatively small and will be chosen later, consider the following variation vector field:

$$Y_i(s) = \begin{cases} sE_i(s), \ s \in [0,1], \\ E_i(s), \ s \in [1, s_0 - r_0], \\ \frac{s_0 - s}{r_0} E_i, \ s \in [s_0 - r_0, s_0]. \end{cases}$$

By standard second variation formula of the minimal geodesic, we have

$$\sum_{i=1}^{n-1} \int_0^{s_0} \left(|\nabla_{\gamma'} Y_i|^2 - R(\gamma', Y_i, \gamma', Y_i) \right) ds \geqslant 0,$$

or

$$0 \leqslant \int_0^1 \left(n - 1 - s^2 \cdot \mathrm{Ric}(\gamma', \gamma') \right) ds + \int_1^{s_0 - r_0} \left(-\mathrm{Ric}(\gamma', \gamma') \right) ds +$$
$$+ \int_{s_0 - r_0}^{s_0} \left(\frac{n-1}{r_0^2} - \left(\frac{s_0 - s}{r_0} \right)^2 \mathrm{Ric}(\gamma', \gamma') \right) ds.$$

We can reformulate it in the following way,

$$\int_0^{s_0 - r_0} \mathrm{Ric}(\gamma', \gamma') ds \leqslant n - 1 + \int_0^1 (1 - s^2) \mathrm{Ric}(\gamma', \gamma') ds + \frac{n-1}{r_0} +$$
$$- \int_{s_0 - r_0}^{s_0} \left(\frac{s_0 - s}{r_0} \right)^2 \mathrm{Ric}(\gamma', \gamma') ds.$$

Using Ricci lower bound on the last term of right hand side, and noticing the uniform bound of the metric in the unit ball around the fixed point O, one arrives at

$$(15) \qquad \int_0^{s_0 - r_0} \mathrm{Ric}(\gamma', \gamma') ds \leqslant C + \frac{n-1}{r_0} + Cr_0.$$

Our next step is to control the integral of $\mathrm{Ric}(\gamma', \gamma')$ between $s_0 - r_0$ and s_0. Let's consider the following two cases.

- **Case 1:** If there exists a uniform constant $D > 0$ (which will eventually be chosen to be large enough in our application), such that the scalar curvature at x satisfies

$$R(x) \leqslant Ds_0,$$

  then following the argument as in [Ni05, Proposition 1.1], we can achieve the desired bound as follows.

Using soliton equation again, we get $\nabla_i R = 2R_{ij}f_j$, hence

$$(16) \qquad |\nabla R|^2 = 4|R_{ij}f_j|^2.$$

By choosing diagonal forms of Ric and g, using the lower bound on Ricci curvature, we have

$$R_{ii} \geqslant -K, \ R + C \geqslant R_{ii}.$$

Hence (16) implies that

$$|\nabla R|^2 \leqslant 4(R+C)^2|\nabla f|^2.$$

It follows that

$$|\nabla \log(R+C)|^2 \leqslant 4|\nabla f|^2 \leqslant Cr^2$$

from previous gradient bound, hence we have $|\nabla \log(R+C)| \leqslant Cr$.

Now for any $s_1 \in [s_0 - r_0, s_0]$, we have

$$\log\left(\frac{R(s_1)+C}{R(s_0)+C}\right) \leqslant \int_{s_1}^{s_0} |\nabla \log(R+C)|ds \leqslant Cs_0(s_0 - s_1).$$

So we arrive at,

$$(17) \qquad R(s_1) + C \leqslant (R(s_0)+C)e^{Cs_0(s_0-s_1)} \leqslant (R(s_0)+C)e^{Cs_0 r_0}.$$

Now let's pick r_0 such that

$$s_0 r_0 = \frac{n-1}{\varepsilon},$$

where ε is a sufficiently small but fixed positive constant. It is clear that $r_0 \leqslant C$ for large s_0.

The integral of Ricci curvature between $s_0 - r_0$ and s_0 can now be controlled as follows,

$$\begin{aligned}
\int_{s_0-r_0}^{s_0} \mathrm{Ric}(\gamma',\gamma')ds &\leqslant \int_{s_0-r_0}^{s_0} (R+C)ds \\
&\leqslant r_0(R(s_0)+C)e^{\frac{C(n-1)}{\varepsilon}} \\
&\leqslant r_0(Ds_0+C)e^{\frac{C(n-1)}{\varepsilon}} \\
&\leqslant C.
\end{aligned}$$

Here we used the assumption on $R(s_0)$ and (17) in the above estimate. Combining this with (15), we show that, for small $\varepsilon > 0$,

$$\int_0^{s_0} \mathrm{Ric}(\gamma',\gamma')ds \leqslant C + \frac{n-1}{r_0} + Cr_0 + C \leqslant C + \varepsilon s_0.$$

Then one can deduce a (convex) quadratic lower bound for $f(x)$ by

$$\gamma'(f)(x) - \gamma'(f)(O) = \int_0^{s_0} \nabla^2(f)(\gamma', \gamma')ds$$

$$= \int_0^{s_0} (\rho - \text{Ric}(\gamma', \gamma'))ds$$

$$\geqslant (\rho - \varepsilon)s_0 - C,$$

which proves that $f(x) \geqslant \frac{\rho}{2}r(x)^2$ for x with large $r(x) = s_0$.

- **Case 2:** If $R(x) \geqslant Ds_0$, then as $R + |\nabla f|^2 - 2\rho f = 0$, we have

$$f = \frac{1}{2\rho}(R + |\nabla f|^2) \geqslant \frac{Ds_0}{2\rho}.$$

In conclusion, we have, for x with large $r(x)$,

$$f(x) \geqslant min\{\frac{\rho}{2}r(x)^2, \frac{Dr(x)}{2\rho}\}.$$

The proof of the lemma is thus finished.

$\square$

Now we can finish the proof of Theorem 1.

Proof of Theorem 1. The first author proved the identity (4) when M is a closed shrinking gradient soliton in [Cao07, P427 $\sim$ P429]. The key point of the proof is using integration by parts (IBP). If we can show that all of the IBPs in [Cao07, P427 $\sim$ P429] are valid for the soliton (M, g) in Theorem 1, then we can repeat the proof.

The integrations by parts we need to check are listed as follows:

(18) $$\int_M R_{lkjp}f_p\nabla_k R_{jl}e^{-f}d\mu = -\int_M R_{lkjp}f_{pk}R_{jl}e^{-f}d\mu ,$$

(19) $$\int_M \nabla_k R_{jl}\nabla_l R_{jk}e^{-f}d\mu = \int_M R_{jk}(\nabla_j R_{ik}f_i - \nabla_i\nabla_j R_{ik})e^{-f}d\mu ,$$

(20) $$\int_M R_{jk}\nabla_j R_{ik}f_i e^{-f}d\mu = -\int_M R_{jk}R_{ik}f_{ij}e^{-f}d\mu ,$$

(21) $$\int_M \nabla_j\nabla_i R_{ik}R_{jk}e^{-f}d\mu = -\int_M \nabla_i R_{ik}\nabla_j(R_{jk}e^{-f})d\mu ,$$

where $f_p = \nabla_p f$, $f_{pk} = \nabla_p\nabla_k f$, etc. We will just check (18). The others are similar.

Before we check the above identities, we need the derivative estimate of curvatures. We now view our shrinking soliton as a solution to the Ricci flow on $t \in [-1, 0]$ with $g(0) = g$ and

(22) $$|R_{ijkl}|(y,t) \leqslant C\exp\left(\frac{3}{2}ar(x)\right) , \quad \forall (y,t) \in B_{g(-1)}\left(x, \frac{r(x)}{2}\right) \times [-1, 0],$$

here $r(x)$ is the distance function to some fixed point $O \in M$ with respect to the metric $g(0)$, C is some constant depends only on n. Applying the local derivative estimate of W.-X. Shi [Shi89] (cf. [Ham95]), we have

$$|\nabla \mathrm{Rm}|(x,0) \leqslant C \exp\left(\frac{9}{4} a r(x)\right),$$

$$|\nabla \nabla \mathrm{Rm}|(x,0) \leqslant C \exp\left(\frac{27}{8} a r(x)\right),$$

etc., here all the constants C depend only on n.

We need to choose D to be large enough in Lemma 6. Clearly the lower bound for $f(x)$ would then be $\frac{D}{2\rho} r(x)$ for large $r(x)$.

Now we turn to our integral identities. Fix a point $O \in M$, and let $B_r = B_r(O)$ be the ball centered at O with radius $r > 0$ in M. Let $X_k = R_{lkjp} f_p R_{jl} e^{-f}$, then

$$R_{lkjp} f_p \nabla_k R_{jl} e^{-f} = \nabla_k X_k - R_{lkjp} f_{pk} R_{jl} e^{-f},$$

where we use the identity (11). Hence we have

(23)
$$\int_{B_r} R_{lkjp} f_p \nabla_k R_{jl} e^{-f} d\mu = \int_{\partial B_r} \langle X, \nu \rangle dA - \int_{B_r} R_{lkjp} f_{pk} R_{jl} e^{-f} d\mu,$$

where ν is the unit normal vector field on ∂B_r and dA is the induced measure on ∂B_r. We claim that $\int_{\partial B_r} \langle X, \nu \rangle dA \to 0$ as $r \to \infty$ and the other integrals in (23) are finite as $r \to \infty$. In fact since $\mathrm{Ric} \geqslant -Kg$, by Bishop-Gromov Comparison Theorem (cf. [SY94]), which gives linear exponential growth control of the volume form, as $r \to \infty$,

$$\left| \int_{\partial B_r} \langle X, \nu \rangle dA \right| \leqslant C \int_{\partial B_r} |\mathrm{Rm}| \cdot |\nabla f| \cdot |\mathrm{Ric}| e^{-f} dA$$

$$\leqslant C \int_{\partial B_1} \exp\left[-\frac{D}{2\rho} r + (2a+C)r\right] dA_1 \to 0,$$

where D is chosen to be large enough. For the other integrals, still using Bishop-Gromov Comparison Theorem, as for any r, we have

$$\left| \int_{B_r \setminus B_1} R_{lkjp} f_p \nabla_k R_{jl} e^{-f} d\mu \right| \leqslant \int_{B_r \setminus B_1} |\mathrm{Rm}| \cdot |\nabla f| \cdot |\nabla \mathrm{Ric}| e^{-f} d\mu$$

$$\leqslant C \int_1^r \exp\left[-\frac{D}{2\rho} t + \left(\frac{13}{4} a + C\right) t\right] dt < \infty,$$

and

$$\left| \int_{B_r \setminus B_1} R_{lkjp} f_{pk} R_{jl} e^{-f} d\mu \right| \leqslant \int_{B_r \setminus B_1} |\mathrm{Rm}| \cdot (|\mathrm{Ric}| + n\rho) \cdot |\mathrm{Ric}| e^{-f} d\mu$$

$$\leqslant C \int_1^r \exp\left[-\frac{D}{2\rho} t + (3a+C)t\right] dt < \infty.$$

This leads to

$$\left| \int_{B_r} R_{lkjp} f_p \nabla_k R_{jl} e^{-f} d\mu \right| < \infty \quad \text{and} \quad \left| \int_{B_r} R_{lkjp} f_{pk} R_{jl} e^{-f} d\mu \right| < \infty .$$

So we verify that (18) is valid. □

Now we will finish the proof of Theorem 3 using the above integral identity.

Proof of Theorem 3. Since $f(x) \leqslant Cr(x)^2$ for x with $r(x)$ large enough, and $R + |\nabla f|^2 = 2\rho f$, so we have

$$R \leqslant 2\rho f \leqslant Cr^2 .$$

With (M, g) is locally conformally flat and the Ricci curvature is bounded from below, it implies that $|\operatorname{Ric}| \leqslant Cr^2$. So the condition of Theorem 1 is satisfied.

If (M, g) is not flat, following the proof of Theorem 2, we have

$$\frac{1}{n} \int_M |\nabla R|^2 e^{-f} d\mu \leqslant \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} d\mu .$$

Hence we have $\nabla R = 0$ and $\nabla \operatorname{Ric} = 0$. Since (M, g) is locally conformally flat, we conclude that $\nabla \operatorname{Rm} = 0$, i.e., M is a locally symmetric space.

Using $\nabla \operatorname{Ric} = 0$ and $\nabla_i (R_{ik} e^{-f}) = 0$, we have

$$0 = (\nabla_i R_{ik}) e^{-f} - R_{ik} f_i e^{-f} = -R_{ik} f_i e^{-f} ,$$

which implies $R_{ik} f_i = 0$ (this identity also follows from the identity $\nabla_i R = 2R_{ij} f_j$ and the fact that R is constant). Moreover, we have

$$0 = \nabla_j (R_{ik} f_i) = R_{ik} f_{ij} , \quad \forall j, k = 1, \dots, n ,$$

here we use the fact that $\nabla \operatorname{Ric} = 0$. For any $x \in M$, we diagonalize the Ricci curvature tensor $\operatorname{Ric}(x)$ in an orthonormal frame (hence the Hessian $\nabla\nabla f$ is also diagonalized because of the soliton equation (2)),

$$R_{ij} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} .$$

So we arrive at $\lambda_i \cdot f_{ii} = 0$ for each $i \in \{1, \dots, n\}$. This implies that, for each i, either

$$\lambda_i = 0,$$

or

$$f_{ii} = 0 \text{ but } \lambda_i \neq 0.$$

When $f_{ii} = 0$, we get $\lambda_i = \rho$ by the soliton equation (2). Therefore without loss of generality, we may assume

$$\lambda_i = \begin{cases} 0 , & 1 \leqslant i \leqslant m , \\ \rho , & m+1 \leqslant i \leqslant n , \end{cases}$$

where $1 \leqslant m \leqslant n$. Now since (M,g) is locally conformally flat, by equation (6), we have

$$R_{ijij} = \frac{1}{(n-1)(n-2)}[(n-1)(\lambda_i + \lambda_j) - (n-m)\rho] \,.$$

Using an identity of Berger (also see Lemma 4.1 in [Cao07]) and the fact that $\nabla \mathrm{Ric} = 0$, we have

$$
\begin{aligned}
0 = \sum_{i<j} R_{ijij}(\lambda_i - \lambda_j)^2 &= \left(\sum_{i<j\leqslant m} + \sum_{i\leqslant m<j} + \sum_{m+1\leqslant i<j} \right) R_{ijij}(\lambda_i - \lambda_j)^2 \\
&= \frac{1}{(n-1)(n-2)} \sum_{i\leqslant m<j} [(n-1)(\lambda_i + \lambda_j) - (n-m)\rho](\lambda_i - \lambda_j)^2 \\
&= \frac{1}{(n-1)(n-2)} m(m-1)(n-m)\rho^3 \,.
\end{aligned}
$$

Thus $m = 0,1$ or n, then we get

- $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \rho$, or
- $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \rho$, or
- $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Since $R = \lambda_1 + \cdots + \lambda_n$ is constant on M, the above result is global (i.e. if the eigenvalues of the Ricci curvature tensor $\lambda_1, \ldots, \lambda_n$ belong to one of the above three cases at one point, then they belong to the same case at any other points).

For the first case, $R = n\rho > 0$, and we have

$$R_{ij} = \frac{R}{n} g_{ij} = \rho g_{ij} > 0 \,,$$

so (M,g) is compact, but we assume (M,g) is complete noncompact, this is a contradiction.

For the second case, $R = (n-1)\rho > 0$, and we have

$$R_{1i} = R_{i1} = 0 \,, \quad 1 \leqslant i \leqslant n \,,$$

and

$$R_{jk} = \frac{R}{n-1} g_{jk} \,, \quad 2 \leqslant j,k \leqslant n \,.$$

For the third case, we have $R_{ij} = 0$ and $R = 0$, so $R_{ijkl} = 0$, i.e., (M,g) is flat. This also yields a contradiction.

Hence we can conclude that the sectional curvature

$$K(e_i, e_j) = \frac{R_{ijij}}{g_{ii}g_{jj} - g_{ij}^2}$$

is nonnegative, so (M,g) is a simply connected locally symmetric space of nonnegative sectional curvature, then we prove the statements of this theorem by using Theorem 10.1.1 in [Wol84]. □

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