Assignment 1

"The hyperbolic plane"

Due Tuesday, 17 January, at the start of the 9:00 lecture. Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Define the *cross ratio* of pairwise distinct points $a, b, c, d \in \widehat{\mathbb{C}}$ (in this order!) by:

$$\mathrm{CR}[a, b, c, d] = \frac{(a - c)(b - d)}{(a - d)(b - c)}.$$

If one of the points is ∞, then the correct expression is obtained by continuous extension. For instance, if $a = \infty$, consider the limiting value as $a \to \infty$, giving $\mathrm{CR}[\infty, b, c, d] = \dfrac{(b - d)}{(b - c)}$.

You have shown in an exercise that for any Möbius transformation $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, we have

$$\mathrm{CR}[M(a), M(b), M(c), M(d)] = \mathrm{CR}[a, b, c, d].$$

Q1 (A formula for hyperbolic distance)

Given $z, w \in \mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, let $z^*, w^* \in \mathbb{R} \cup \{\infty\}$ be the endpoints of the complete geodesic L passing through z and w, in such a way that z^*, z, w, w^* occur in this order on L. Show that the hyperbolic distance between z and w satisfies

$$d(z_1, z_2) = \log \mathrm{CR}[z, w, w^*, z^*].$$

Q2 (Möbius transformations)

Suppose $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Möbius transformation and $p_1, p_2, p_3 \in \widehat{\mathbb{C}}$ are the three points satisfying $M(p_1) = 1$, $M(p_2) = 0$ and $M(p_3) = \infty$. Show that for every $z \in \widehat{\mathbb{C}} \setminus \{p_1, p_2, p_3\}$, we have

$$M(z) = \mathrm{CR}[z, p_1, p_2, p_3].$$

Q3 (Neighbourhoods)

Let $L = \{iy \mid y > 0\}$ be the complete geodesic in the upper half-plane with ideal endpoints 0 and ∞. Given $r > 0$, show that the locus of points of hyperbolic distance r from L consists of the two euclidean lines that meet L at 0 in the angle ϑ, where

$$\frac{1}{\cos \vartheta} = \cosh r.$$

Hint: If $q \notin L$, then the point on L closest to q lies on the complete geodesic passing through q and meeting L at a right angle (i.e. the geodesic is a semi-circle with centre at 0).

Q4 (Fixed points and trace of commutator)

Let $A, B \in SL_2(\mathbb{C})$ be distinct from $\pm E$, so that the Möbius transformations $z \mapsto A \cdot z$ and $z \mapsto B \cdot z$ are non-trivial.

(a) The transformations share a fixed point if and only if $\mathrm{tr}(ABA^{-1}B^{-1}) = +2$.

(b) Suppose the transformations do not share a fixed point. Then $ABA^{-1}B^{-1}$ is parabolic or equal to $-E$ if and only if $\mathrm{tr}(ABA^{-1}B^{-1}) = -2$.

Hint for (a): You may suppose w.l.o.g. that one of the transformations is in standard form.