# Topology & Groups

## Michaelmas 2008

## Question Sheet 6

Questions with an asterisk $*$ beside them are primarily aimed at 4th year students.

1. Recall that $G * H$ denotes the free product of groups $G$ and $H$. Let $\alpha\colon G \to G * H$ be one of the canonical homomorphisms. Find a homomorphism $\pi\colon G * H \to G$ such that $\pi\alpha = \mathrm{id}_G$. Deduce that $\alpha$ is injective.

$*$ 2. Any element of $G * H$ is represented by a word in the alphabet $G \cup H$. We may perform the following operations to such a word, without changing the element of $G * H$ that it represents:

   (I) if successive letters $g_1$ and $g_2$ belong to $G$ (or they both belong to $H$), then amalgamate them to form the letter $g_3$, where $g_3 = g_1 g_2$ in $G$ (or $H$);

   (II) if some letter is the identity in $G$ or $H$, remove it.

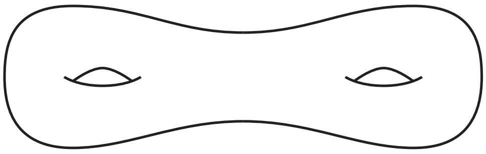
   Each of these operations shortens the word, and so eventually we will reach a stage a where they cannot be performed any further. The resulting word is $g_1 h_1 g_2 h_2 \dots g_n h_n$, where $g_i \in G$ and $h_i \in H$, and each $g_i$ and each $h_i$ is non-trivial, except possibly $g_1$ and/or $h_n$. We then say that this word is *reduced*. Prove that each element of $G * H$ has a *unique* reduced representative. [Hint: emulate the proof of IV.8 by formulating and proving a suitable version of IV.9.]

3. (i) Let $T$ be the torus, which is obtained from the square by the usual side identifications. Let $D$ be a small open disc at the centre of the square. Let $X$ be the space obtained from $T$ by removing $D$. Let $\partial D$ be the boundary curve of $D$, and let $b$ be a basepoint on $\partial D$. Prove that $\pi_1(X, b)$ is isomorphic to a free group on two generators.

   (ii) What word in these generators does the loop $\partial D$ spell?

   (iii) Now let $S$ be the surface 'with two handles', as shown on the following page. Show that $S$ can be obtained by taking two copies of $X$ and gluing them along the two copies of $\partial D$.

   (iv) Deduce that $\pi_1(S)$ is an amalgamated free product.



4. Construct simply-connected covering spaces of the following spaces:

   (i) the Möbius band,

   (ii) $S^2 \vee S^1$,

   (iii) $\mathbb{R}^2 - \{\text{point}\}$.