Math3402 Problem Set 8

Question 1: Consider two non-negative numbers a_0 and b_0. Define sequences a_n and b_n by

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

i) What does the AM-GM inequality tell us about a_n and b_n?
ii) Using part *i)*, Show that a_n is a non-increasing sequence bounded below, while b_n is a non-decreasing sequence bounded above.
iii) Explain why this means a_n and b_n are convergent sequences, and show that their limit $M(a_0, b_0)$ is the same. $M(a_0, b_0)$ is called the arithmetic-geometric mean of a_0 and b_0.
iv) Explain why $M(a_0, b_0) = M(\frac{a_0+b_0}{2}, \sqrt{a_0 b_0})$.

Question 2: Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for $a, b > 0$, use Jensen's inequality to show that for any $\varepsilon > 0$,

$$ab \le \varepsilon \frac{a^p}{p} + \varepsilon^{-q/p} \frac{b^q}{q}.$$

Question 3: *i)* Show that a norm is a convex function.
ii) Show that for any normed space $(V, \|\cdot\|)$, the closed unit ball

$$\overline{B}_1(0) = \{x \in V : \|x\| \le 1\}$$

is a convex set.

Question 4: Show that the following are norms on $\mathbb{R}^2$, and in each case sketch the closed unit ball $\overline{B}_1(0)$:
i) $\|x\|_1 = |x| + |y|$
ii) $\|x\|_2 = (x^2 + y^2)^{1/2}$
iii) $\|x\|_\infty = max\{|x|, |y|\}$

Question 5: Let V be a normed space and $W \subset V$ a subspace. Show that the closure $\overline{W}$ of W is also a subspace.