Problem Set 2

Q09 Let (X, d) be a metric space, and let $A \subseteq X$. Show that if the limit of each convergent sequence in A is also in A, then A is closed. (This is the reverse implication of 1.16).

Q10 Let (X, d) be a metric space. Show that $A \subseteq X$ is bounded if and only if there are $x \in X$ and $R > 0$, such that $A \subseteq B_R(x)$.

Q11 (a) A closed subset of a complete metric space is complete.
(b) A complete subset of an arbitrary metric space is closed.

Q12 [Puzzle] Find a statement using closed sets that is equivalent to the Heine-Borel property.

Q13 Two metrics d_1 and d_2 on the non-empty set X are *equivalent* if there are constants $a, b > 0$, such that

$$a\ d_1(x, y) \leq d_2(x, y) \leq b\ d_1(x, y)$$

for all $x, y \in X$. Prove the following statements:

(a) The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent (resp. Cauchy) with respect to d_1 if and only if it is convergent (resp. Cauchy) with respect to d_2.
(b) The subset $A \subseteq X$ is open (resp. closed, compact) with respect to d_1 if and only if it is open (resp. closed, compact) with respect to d_2.
(c) The function $f\colon X \to Y$, where Y is a metric space, is continuous (resp. uniformly continuous) with respect to d_1 if and only if it is continuous (resp. uniformly continuous) with respect to d_2.

Q14 Let (X, d) be a compact metric space. Show that for every open cover $U = \{U_\alpha \mid \alpha \in I\}$ there exists $\varepsilon > 0$, such that for each $x \in X$ there is $\alpha \in I$ such that $B_\varepsilon(x) \subseteq U_\alpha$. The number ε is called a *Lebesgue number for U*.

Q15 Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Prove that d is a metric on $X \times Y$. This is called the *product metric* on $X \times Y$.

Q16 Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \min\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Is d is a metric on $X \times Y$?

Q17 Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Is d is a metric on $X \times Y$?

Q18 Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

Is d is a metric on $X \times Y$?

Q19 Show that the Euclidean metric on $\mathbb{R}^2$ and the product metric on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ inherited from the Euclidean metric on $\mathbb{R}$ (see Q15) are equivalent.