Math3402 Assignment 5

Due Thursday May 12th

Question 1: Suppose $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$, and let $x(t)$ be a differentiable path in $\mathbb{R}^n$. Show that

$$\frac{d}{dt}\|x(t)\| \le \|\frac{d}{dt}x(t)\|,$$

when the left-hand-side exists. Explain why the limit can be placed inside the norm.

Question 2: Show that a normed space X is homeomorphic to the open ball $B_1(0) = \{x \in X : \|x\| < 1\} \subset X$. Hint: Consider the mapping $x \to \frac{1}{1+\|x\|}x$.

Question 3: Let $B(X, X)$ be the space of bounded linear operators from X to itself and let S be the subset consisting of the invertible operators. Show that S is open in the topology induced by the operator norm.

Question 4: Let T be an element of $B(X, X)$ (the bounded linear operators from X to X), and let W be a subset of X. Show that $T(\overline{W}) \subseteq \overline{T(W)}$. Show furthermore that if T has bounded inverse, then in fact $T(\overline{W}) = \overline{T(W)}$.