Assignment 3

Due Thursday, 7 April, at 17:00 in the assignment box for MATH3402. The box is located on **Level 3** in the Mathematics (Priestley) building (67). It is in the **white lot of boxes** (there are two lots, to find ours, turn left as you come from the stairs).

Please use a cover sheet!

Clearly state your assumptions and conclusions, and justify all steps in your work.

Marks will be deducted for sloppy working.

Q1 (a) Let $(X, \mathcal{O})$ be a topological space, and give $X \times X$ the product topology. Show that X is Hausdorff if and only if the diagonal $\{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$.

(b) Give an example of a non-Hausdorff space $(X, \mathcal{O})$ and verify that the diagonal in $X \times X$ is not closed with respect to the product topology.

Q2 Let $(X, \mathcal{O})$ be a topological space and $\sim$ be an equivalence relation on X. Denote $\overline{X}$ the set of all equivalence classes and $p\colon X \to \overline{X}$ the map taking each $x \in X$ to its equivalence class, i.e.

$$p(x) = [x] = \{y \in X \mid y \sim x\}.$$

Let $\mathcal{Q} = \{U \subseteq \overline{X} \mid p^{-1}(U) \in \mathcal{O}\}$.

(a) Show that $\mathcal{Q}$ is a topology on $\overline{X}$. This is called the *quotient topology.*

(b) Show that $p\colon X \to \overline{X}$ is continuous with respect to $\mathcal{O}$ and $\mathcal{Q}$.

(c) Show that $(\overline{X}, \mathcal{Q})$ is T_1 if and only if every equivalence class is a closed subset of $(X, \mathcal{O})$.

Q3 Let G be one of the following sets of matrices:

(a) $\left\{ \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \mid \vartheta \in \mathbb{R} \right\}$

(b) $\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\}$

(c) $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}$

You may assume that for each choice of G, an equivalence relation on $\mathbb{R}^2$ is defined by:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \sim \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \iff \exists M \in G \text{ such that } M \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Let $(X, \mathcal{O}) = (\mathbb{R}^2, \mathcal{O}_{\mathbb{E}})$, and let $(\overline{X}, \mathcal{Q})$ be the quotient space under the above equivalence relation with the quotient topology. Determine the following facts about the respective quotient topologies for the three choices of G:

(a) $\overline{X}$ is homeomorphic with $[0, \infty)$, where the latter has the subspace topology from $(\mathbb{R}, \mathcal{O}_{\mathbb{E}})$;

(b) $\overline{X}$ is T_1 but not Hausdorff (T_2);

(c) $\overline{X}$ is T_0 but not T_1.