Problem Set 4

Q1 Show that two equivalent norms on the same vector space induce the same topology.

Q2 Show that if $(X, ||\cdot||_0)$ is complete and $||\cdot||_1$ equivalent to $||\cdot||_0$, then $(X, ||\cdot||_1)$ is complete.

Q3 (a) Find two normed spaces that are algebraically isomorphic, but not topologically isomorphic.

(b) Find two normed spaces that are topologically isomorphic, but not isometrically isomorphic.

Q4 How is the Banach–Mazur distance between X and Y related to the Banach–Mazur distance between X^* and Y^*?

Q5 Let X be a topological space.

(a) Show that $Y \subseteq X$ is dense if and only if for each non-empty, open subset $U \subseteq X$: $Y \cap U \neq \emptyset$.

(b) Show that the intersection of finitely many open and dense sets in X is open and dense in X.

Q6 Show that Version 2 of the Baire Category Theorem implies Version 1.

Q7 Let X be a topological space and $Y \subseteq X$. Show that the following are equivalent:

(a) Y is no-where dense in X.

(b) $X \setminus \overline{Y}$ is open and dense in X.

(c) $X \setminus Y$ contains an open and dense set.

(d) $X \setminus \overline{(X \setminus \overline{Y})} = \emptyset$

(e) $Y \subset \overline{(X \setminus \overline{Y})}$

Q8 Let $B = B(l_1)$. Show that B as a subset of l_2 is closed and has empty interior. Conclude that l_1 as a subset of l_2 is meagre and that

$$\{x \in l_2 \mid \sum |x_i| = \infty\}$$

is dense in l_2.

Q9 Let X be a normed space and $S \subseteq X$. Show that if $\{f(x) \mid x \in S\}$ is bounded for each $f \in X^*$, then S is bounded.