HAMILTONIAN SPECTRAL FLOWS, THE MASLOV INDEX, AND THE STABILITY OF STANDING WAVES IN THE NONLINEAR SCHRODINGER EQUATION[*]

GRAHAM COX[†], MITCHELL CURRAN[‡], YURI LATUSHKIN[§], AND ROBERT MARANGELL[¶]

Abstract. We use the Maslov index to study the spectrum of a class of linear Hamiltonian differential operators. We provide a lower bound on the number of positive real eigenvalues, which includes a contribution to the Maslov index from a nonregular crossing. A close study of the eigenvalue curves, which represent the evolution of the eigenvalues as the domain is shrunk or expanded, yields formulas for their concavity at the nonregular crossing in terms of the corresponding Jordan chains. This enables the computation of the Maslov index at such a crossing via a homotopy argument. We apply our theory to study the spectral (in)stability of standing waves in the nonlinear Schrödinger equation on a compact interval. We derive stability results in the spirit of the Jones–Grillakis instability theorem and the Vakhitov–Kolokolov criterion, both originally formulated on the real line. A fundamental difference on passing from the real line to the compact interval is the loss of translational invariance, in which case the zero eigenvalue of the linearized operator is (typically) geometrically simple. Consequently, the stability results differ depending on the boundary conditions satisfied by the wave. We compare our lower bound to existing results involving constrained eigenvalue counts, finding a direct relationship between the correction factors found therein and the objects of our analysis, including the second-order Maslov crossing form.

Key words. Maslov index, Hamiltonian systems, nonlinear Schrodinger equation, eigenvalues, differential operators, stability

MSC codes. 53D12, 34L40, 37K45, 47A75

DOI. 10.1137/22M1533797

1. Introduction. We use the Maslov index to study the real spectrum of Hamiltonian differential operators of the form

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix},$$

where $L_\pm$ are scalar-valued Schrödinger operators with arbitrary C^2 potentials on a compact interval $[0, \ell]$. In particular, we provide a lower bound on the number of positive real eigenvalues of the operator N (Theorem 2.2).

Our approach is to restrict N to a subinterval $[0, s\ell]$, $s \in (0, 1]$ and, rescaling back to $[0, \ell]$, study the s-dependent spectrum of the one-parameter family of operators in the spatial parameter s. We are thus led to a characterization of the eigenvalues

†Memorial University, St. John's A1C 5S7, NL, Canada (gcox@mun.ca).
‡School of Mathematics and Statistics, University of Sydney, Sydney 2006, NSW, Australia (mcur3925@uni.sydney.edu.au).
§Mathematics, University of Missouri, Columbia, MO 65211 USA (latushkiny@missouri.edu).
¶School of Mathematics and Statistics F07, University of Sydney, Sydney 2006, NSW, Australia (robert.marangell@sydney.edu.au).

of the rescaled operators as a locus of points in the λs-plane (with λ the spectral parameter), which we refer to as *eigenvalue curves*. We interpret the eigenvalue curves as loci of intersections, or *crossings*, of a path in the manifold of Lagrangian planes with a certain codimension-one subvariety. This affords the use of the Maslov index, a signed count of such crossings. Formulas for the concavity of the eigenvalue curves are given (Theorems 2.9, 4.5, and 4.6), and are used to compute a correction term appearing in the lower bound in Theorem 2.2.

Operators of the form of N arise in the linearization about a standing wave solution $\widehat{\psi}(x,t) = e^{i\beta t}\phi(x)$ of the nonlinear Schrödinger (NLS) equation

$$(1.1) \qquad\qquad i\psi_t = \psi_{xx} + f\left(|\psi|^2\right)\psi,$$

where $\psi : [0,\ell] \times [0,\infty) \longrightarrow \mathbb{C}$, the nonlinearity $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a C^3 function, and $\beta \in \mathbb{R}$ is the temporal frequency. The wave around which we linearize is said to be *spectrally unstable* if there exists spectrum of N in the open right half plane and *spectrally stable* otherwise. By applying Theorem 2.2, we establish stability criteria for standing waves in the NLS equation on a compact interval subject to perturbations satisfying Dirichlet boundary conditions. Namely, we derive analogues of the *Jones–Grillakis instability theorem* (Corollary 2.7) and the *Vakhitov–Kolokolov (VK) criterion* (Theorem 2.11). While Corollary 2.7 is also a consequence of the abstract result of [52, Theorem 3.2], Theorem 2.11, which makes use of the concavity formulas of Theorem 2.9, appears to be new for the case of the compact interval. These two stability results actually remain valid for a spatially dependent nonlinearity $f(x,|\psi|^2)$; see Remark 2.6.

Along the way, we find *Hadamard-type* formulas for the slope of the eigenvalue curves as the ratio of certain quadratic forms, called *crossing forms,* whose signatures locally determine the Maslov index (Proposition 4.2 and Corollary 4.4). Variational formulas for the eigenvalues of boundary value problems with respect to perturbation of the domain are classical and go back to the work of Hadamard [36], Rayleigh [67], and Rellich [68]; see also [38, 35] and [54, section VII.6.5]. Recently, such formulas have been given in terms of the (Maslov) crossing form for families of Schrödinger [56, 59] and abstract self-adjoint operators [58]. Our formulas agree with and build on those found therein.

We also encounter a *nonregular* crossing when $\lambda = 0$, corresponding to a degeneracy of the associated crossing form and points of zero slope for the eigenvalue curves. Geometrically, this corresponds to the Lagrangian path tangentially intersecting the relevant codimension-one subvariety. Some care is then required in order to compute the Maslov index, and it is a key feature of the current work that we are able to do so (Theorem 4.14). In particular, it is sufficient to know the concavity of the eigenvalue curve through the nonregular crossing as well as whether the operators L_+ and L_- have a nontrivial kernel. To the best of our knowledge, no such computation has previously been made in the literature. Analyzing the nonregular crossing in the context of the NLS equation leads to stability criteria that resemble the VK criterion in certain cases, furnishing an interesting connection between the concavity of the eigenvalue curve at the nonregular crossing, the Maslov index there, and the classical VK result; see section 5.

In the case when the spatial domain is the entire real line, if zero is a hyperbolic fixed point of the standing wave equation

$$(1.2) \qquad\qquad \phi_{xx} + f(\phi^2)\phi + \beta\phi = 0$$

and there exists an orbit that is homoclinic to it in the phase plane, a localized solution to (1.1) exists and belongs to $L^2(\mathbb{R})$ for all time. In this case, L_+ and L_-, which are

unbounded operators on $L^2(\mathbb{R})$, both have a nontrivial kernel. Indeed, the stationary state ϕ and its derivative ϕ_x satisfy $L_-\phi = 0$ (the stationary equation (1.2)) and $L_+\phi_x = 0$ (the associated variational equation), respectively, and decay exponentially as $x \to \pm\infty$. By the results of Jones [48] and Grillakis [32], one then has that if $P - Q \neq 0, 1$, where P and Q are the numbers of negative eigenvalues (or *Morse indices*) of L_+ and L_-, then N has at least one positive real eigenvalue, and hence the standing wave solution to (1.1) is unstable. In the edge case when $P = 1$ and $Q = 0$, the results of Vakhitov and Kolokolov [74] and Grillakis, Shatah, and Strauss [33, 34] dictate that the wave is spectrally (and orbitally) stable if the β-derivative of the mass of the wave

$$(1.3) \qquad\qquad \frac{\partial}{\partial\beta} \int_{-\infty}^{\infty} \phi^2 \, dx$$

is negative and spectrally unstable if (1.3) is positive (see [66, Theorem 4.4, page 215]).

One of the key differences on passing from the real line to the compact interval is that, generically, the operators L_+ and L_- (equipped with Dirichlet boundary conditions) do not simultaneously have a nontrivial kernel. Depending on the boundary conditions satisfied by the wave profile ϕ, typically zero will lie in the spectrum of either L_+ or L_- (or neither). A physical reason for this is the loss of translational invariance, which manifests in the failure of the relevant boundary conditions of arbitrary translates of ϕ. As a consequence, our stability results (Corollary 2.7 and Theorem 2.11) will differ depending on which of the operators $L_\pm$ has a nontrivial kernel. In the case that L_- has a nontrivial kernel, we can recover the integral expression (1.3) appearing in the classical VK criterion. Such a recovery is not possible when L_+ has a nontrivial kernel; for details, see the discussion in subsection 5.3.2.

There is a large body of work relating the Morse index of a self-adjoint operator and its number of conjugate points (which was later interpreted as the Maslov index of an associated Lagrangian path), going back to the middle of last century [1, 2, 6, 24, 25, 73]. Most of these theorems can be viewed as generalizations of the classical Sturmian theory, and indeed in [6, 25, 73], they are framed as such, where the nodal count of an eigenfunction indicates where in the sequence of eigenvalues the corresponding eigenvalue sits. Following on from Jones's seminal work [48], the idea of using the Maslov index for spatially Hamiltonian systems to extrapolate temporal spectral information has proven quite fruitful in the ensuing years (see, for example, [49, 18, 19, 43, 42, 57] and the references therein for a partial list of results).

In more recent times, Deng and Jones in [23] (see also [18, 19]) used the Maslov index to analyze second-order elliptic eigenvalue problems on bounded domains. An important feature of this analysis, as well as that of [4, 43, 42, 44, 40], is monotonicity of the Maslov index in the spectral parameter. Monotonicity also holds in the spatial parameter under certain boundary conditions [18, 41, 49]. This property is convenient since it enables an *equality* of the Morse index with the Maslov index of the Lagrangian path corresponding to $\lambda = 0$. Importantly, as in [48], we do *not* have monotonicity in either the spatial or the spectral parameter. However, the signature of crossings in the s-direction when $\lambda = 0$ can always be accounted for, and, consequently, a nonzero Maslov index can nonetheless be used to detect a real, unstable eigenvalue, just as in [61, 62, 47, 71]. This lack of monotonicity thus leads to the *inequality* in Theorem 2.2.

Another feature in the aforementioned references, as well as in [5, 12, 13, 9, 10, 11, 15, 16, 17, 39], is a dynamical systems approach to eigenvalue problems. In these works, the eigenvalue equations associated with the linearized operators are

Hamiltonian or can be made Hamiltonian under a suitable change of variables. The critical feature of such systems is that they induce a symplectically invariant flow and hence preserve the manifold of Lagrangian planes, which affords the application of the Maslov index. For recent works where the Hamiltonian requirement is relaxed, see [15, 16, 17]. In [16, 17], a change of variables is used to recover the Hamiltonian structure, and in [15], the system, while not Hamiltonian, still preserves the space of Lagrangian planes. For an example of where the Hamiltonian requirement is dropped altogether, see [3].

Existing results on the stability of standing wave solutions of (1.1) on a compact spatial interval have been given for periodic solutions of (1.2), with (quasi)periodic perturbations and predominantly for cubic focusing $(f(\phi^2) = \phi^2)$ or defocusing $(f(\phi^2) = -\phi^2)$ NLS. Rowlands in [70] studied the spectral stability of spatially periodic elliptic solutions to the cubic NLS, subject to long wavelength disturbances. Pava [65] showed that the Jacobi dnoidal solutions to cubic focusing NLS were orbitally stable with respect to coperiodic perturbations. In [26], Gallay and Hărăgus showed the orbital stability of spatially periodic and quasiperiodic traveling waves with a complex-valued profile for small-amplitude solutions in both the focusing and the defocusing case. They extended this result to waves of arbitrary amplitude in [27]. For the real-valued (cnoidal) waves, their orbital stability result is restricted to perturbations that are antiperiodic on a half period. This latter condition was done away with in [46], wherein Ivey and Lafortune undertook a spectral stability analysis of the cnoidal traveling wave solutions of the focusing NLS, showing stability with respect to coperiodic perturbations. In [7, 28], the authors extend the orbital stability results for both real- and complex-valued wave profiles to the class of subharmonic perturbations (i.e., perturbations with period an integer multiple of the period of the wave profile) in the defocusing case. In [21, 22], the authors examine the spectral stability of the elliptic solutions with respect to subharmonic perturbations in the focusing case. Unlike the above works, we are interested in the spectral stability of real-valued solutions of (1.2), for an arbitrary C^3 nonlinearity f, that are subject to perturbations satisfying Dirichlet boundary conditions. Moreover, as previously stated, many of our results hold for a spatially dependent f.

Our theory can be extended in several possible directions. In particular, our theory should hold for the case of quasi-periodic boundary conditions on the perturbations, which is natural to consider given that many of the solutions ϕ to (1.2) that satisfy Dirichlet boundary conditions are periodic. The Maslov index has already been used to develop eigenvalue counts for self-adjoint matrix-valued Schrödinger operators with such boundary conditions in [49, 50]. Our theory should also hold when the Schrödinger operators $L_\pm$ are self-adjoint and matrix-valued, and indeed in sections 3 and 4, many of our results are stated for the operator N with an n-dimensional kernel to accommodate this scenario. Finally, while the analysis is significantly more involved, it should be possible to extend to the case where the spatial domain is multidimensional, as in [19, 18, 20].

The paper is organized as follows. In section 2, we set up the eigenvalue problem and state the main results. In section 3, we provide background material on the Maslov index, interpret the (real) eigenvalue problem symplectically, and prove Theorem 2.2. In section 4, we analyze the eigenvalue curves. After computing formulas for their derivatives and relating these to the Maslov crossing forms (Proposition 4.2 and Corollary 4.4), we compute their concavities at the zero eigenvalue (Theorems 4.5 and 4.6), facilitating the computation of the Maslov index at the nonregular crossing (Theorem 4.14). We conclude the section by confirming that the signature of the

second-order Maslov crossing form provides the correct contribution to the Maslov index at this crossing, which is consistent with [23]. In section 5, we provide some applications of Theorems 2.2 and 2.9. In particular, we prove Corollaries 2.7 and 2.8 and Theorem 2.11. We also compute expressions for the concavity (at $s = 1$) of the eigenvalue curve passing through $(\lambda, s) = (0, 1)$ for linearized NLS in each of the cases when L_+ and L_- has a nontrivial kernel (Propositions 5.3 and 5.7). In the latter case, we recover a compact-interval analogue of the classical VK criterion. We conclude the paper with a comparison of the lower bound in Theorem 2.2 with existing results which make use of constrained eigenvalue counts. We find that the "correction" terms appearing in our lower bound and others in the literature are equivalent (Proposition 5.11), applying our formulas to provide new versions of the Hamiltonian–Krein index theorem in terms of the Maslov index (Proposition 5.12).

Notation: We let I_n and 0_n denote the $n \times n$ identity and zero matrices, respectively. We denote the canonical $2n \times 2n$ symplectic matrix and the first Pauli matrix by

$$(1.4) \qquad J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. We let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the L^2 inner product and norm, respectively. Subscript s or λ will indicate dependence of a quantity on these parameters (not derivatives). The spectrum of a linear operator T will be denoted by $\mathrm{Spec}(T)$ and its kernel by $\ker(T)$.

2. Setup and statement of main results. The basic setup is an eigenvalue problem of the form

$$(2.1) \qquad N \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u(\ell) \\ v(\ell) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where N is given by

$$(2.2) \qquad N := \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}$$

and $L_\pm$ are the Schrödinger operators

$$(2.3) \qquad L_+ = -\partial_{xx} - g(x), \qquad L_- = -\partial_{xx} - h(x),$$

with g and h arbitrary functions in $C^2([0, \ell], \mathbb{R})$. To be precise, we consider N as an unbounded operator in $L^2(0, \ell) \times L^2(0, \ell)$ with dense domain

$$(2.4) \quad \mathrm{dom}(N) = \big(H^2(0, \ell) \cap H_0^1(0, \ell) \big) \times \big(H^2(0, \ell) \cap H_0^1(0, \ell) \big) \subset L^2(0, \ell) \times L^2(0, \ell).$$

Hereafter, we drop the product notation on the relevant spaces; it will be clear from the context whether the functions are scalar- or vector-valued. An *eigenvalue* of N is thus a value of $\lambda \in \mathbb{C}$ for which there exists a nontrivial solution $\mathbf{u} := (u, v)^\top$ to the boundary value problem (2.1). Eigenvalues for the unbounded operators $L_\pm$, with dense domains

$$(2.5) \qquad \mathrm{dom}(L_\pm) = H^2(0, \ell) \cap H_0^1(0, \ell) \subset L^2(0, \ell),$$

are similarly defined. Note that the unbounded operators $L_\pm = L_\pm^*$ with domain (2.5) are self-adjoint, while N is not.

Remark 2.1. Notationally, we will not distinguish between the formal differential expressions N and $L_\pm$ and the unbounded operators with domains (2.4) and (2.5)

whose spectra we wish to study. It will be clear from the context in what sense we refer to these objects.

While it is possible for N to have complex eigenvalues, we will restrict our analysis of (2.1) to the case when λ is real and positive. The existence of such an eigenvalue implies instability. On the other hand, there are cases where the spectrum of N lies entirely on the real and imaginary axes, in which case the absence of a real positive eigenvalue implies stability; see Theorem 2.11 for an example.

Our first result is a lower bound for the number of positive real eigenvalues of N. It follows from an application of the Maslov index. The idea is to study the spectral problem in (2.1) via a rescaling of the domain. We restrict (2.1) to a family of subdomains $[0, s\ell]$ using a parameter $s \in (0, 1]$,

$$(2.6) \qquad N\mathbf{u} = \lambda\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}(s\ell) = 0,$$

and define a *conjugate point* to be a value of s for which there exists a nontrivial solution to (2.6) with $\lambda = 0$. We then deduce the existence of unstable eigenvalues of (2.1) by counting conjugate points (via the Maslov index) as s varies from 0 to 1. Defining the quantities

$$P := \#\{\text{negative eigenvalues of } L_+\},$$
$$Q := \#\{\text{negative eigenvalues of } L_-\},$$
$$n_+(N) := \#\{\text{positive real eigenvalues of } N\},$$

we have the following.

THEOREM 2.2. *Let N be an operator as in* (2.2)–(2.3). *The number of positive real eigenvalues of N satisfies*

$$(2.7) \qquad n_+(N) \geq |P - Q - \mathfrak{c}|,$$

where $\mathfrak{c}$ (given in Definition 3.14*) is the total contribution to the Maslov index in the s and λ directions from the conjugate point at $s = 1$. (If there is no such conjugate point, $\mathfrak{c} = 0$.)*

Remark 2.3. One of the main results of this paper is that we are able to give explicit formulas for this so-called corner term $\mathfrak{c}$, which has the property that $\mathfrak{c} \in \{-1, 0, 1\}$. The name derives from the location of the associated crossing in terms of the so-called *Maslov box*. For precise statements, see sections 3 and 4, in particular Theorem 4.14.

Remark 2.4. In (2.6), the symbol N denotes a differential expression. For the associated unbounded operator, we define

$$(2.8) \qquad N|_{[0,s\ell]}\mathbf{u} := N\mathbf{u}, \qquad \mathbf{u} \in \text{dom}(N|_{[0,s\ell]}) = H^2(0, s\ell) \cap H_0^1(0, s\ell) \subset L^2(0, s\ell),$$

so that $\lambda \in \text{Spec}(N|_{[0,s\ell]})$ if and only if (2.6) has a nontrivial solution.

Theorem 2.2 (the proof of which is given in subsection 3.4) is in the spirit of a number of lower bounds in the literature. In contrast to [45, Assumption 2.1(b)], we do not assume that the operators $L_\pm$ are invertible. If both L_+ and L_- are invertible, it will follow that there is no conjugate point at $s = 1$, and therefore $\mathfrak{c} = 0$. In this case, we recover the inequality in [45, Theorem 2.25]. The lower bound for $n_+(N)$ in the case when one or both of L_+ and L_- has a nontrivial kernel has

been studied in [52, Theorem 3.2], [55, Theorem 5.6], [60, Theorem 2.3], and [32, Theorem 1.2], to name a few; see also [53, section 7.1.3]. In these works, the authors typically project off the kernels of L_+ and L_- and give the lower bound in terms of the associated constrained eigenvalue counts for L_+ and L_-. By contrast, we require no such projections. The constrained counts for L_+ and L_- (given in the current work in (5.31)) involve the number of negative eigenvalues of certain matrices denoted $D_\pm$. In subsection 5.4, we will show that our "correction" factor—given by the corner term $\mathfrak{c}$—is equivalent to the "correction" factor in [53, Theorem 7.1.16], given by the difference $n_-(D_+) - n_-(D_-)$ of negative indices of D_+ and D_- (see Proposition 5.11). Thus, Theorem 2.2 together with Proposition 5.11 recovers [53, Theorem 7.1.16]. The Maslov index interpretation afforded by $\mathfrak{c}$ is convenient because it provides a way of *computing* the difference $n_-(D_+) - n_-(D_-)$. Namely, (5.38) shows that the signs of $D_\pm$ (which in our setup are scalars) are given by the signs of the concavities of the eigenvalue curves at $(\lambda, s) = (0, 1)$.

Our main application will be to the linearization of (1.1) about a *standing wave* solution. This is a solution to (1.1) of the form $\widehat{\psi}(x, t) = e^{i\beta t}\phi(x)$ for some $\beta \in \mathbb{R}$, where the real-valued wave profile or *stationary state* $\phi : [0, \ell] \to \mathbb{R}$ solves the time-independent equation

$$(2.9) \qquad \phi_{xx} + f(\phi^2)\phi + \beta\phi = 0.$$

The results of this paper hold under fairly general boundary conditions on ϕ. Two examples that we will often focus on are Dirichlet conditions

$$(2.10) \qquad \phi(0) = \phi(\ell) = 0$$

or Neumann conditions

$$(2.11) \qquad \phi'(0) = \phi'(\ell) = 0.$$

In these cases, one possible choice for the interval length ℓ is to fix a T-periodic solution to (2.9) and to set $\ell = kT/2$ for some $k \in \mathbb{N}$. Some example phase portraits for (2.9) featuring periodic orbits are given in Figure 1. As an aside, note that the
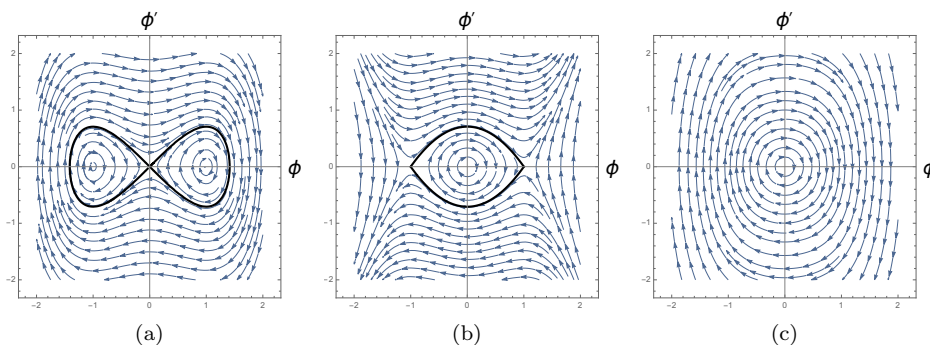

FIG. 1. *Examples of phase portraits for* (2.9). *In* (a), *we have cubic focusing nonlinearity* $f(\phi^2) = \phi^2$ *and* $\beta < 0$. *The homoclinic orbits in black, representing localized solutions on* $\mathbb{R}$, *separate those inside (nonzero Jacobi dnoidal functions) and those outside (Jacobi cnoidal functions that oscillate evenly about* $\phi = 0$). *In* (b), *we have cubic defocusing nonlinearity* $f(\phi^2) = -\phi^2$ *and* $\beta > 0$, *with periodic orbits existing only inside the heteroclinic cycle in black. In* (c), *we have* $f(\phi^2) = \phi^2$ *and* $\beta > 0$.

homoclinic orbits in Figure 1(a) correspond to strictly positive or negative localized solutions on $\mathbb{R}$.

A natural question to ask is whether the standing wave $\widehat{\psi}$ is stable in time with respect to small perturbations in ϕ. Substituting the perturbative solution

$$\psi(x,t) = e^{i\beta t}\left[\phi(x) + \varepsilon e^{\lambda t}(u(x) + iv(x))\right]$$

into (1.1) and collecting $O(\varepsilon)$ terms, we arrive at the differential equations in (2.1), where

$$(2.12) \qquad \begin{aligned} g(x) &= 2f'(\phi^2(x))\phi^2(x) + f(\phi^2(x)) + \beta, \\ h(x) &= f(\phi^2(x)) + \beta. \end{aligned}$$

Then, subject to the class of perturbations $\mathbf{u} = (u,v)^\top$ that vanish at both endpoints, the standing wave $\widehat{\psi}$ is spectrally stable if the spectrum of the linearized operator N is contained in the imaginary axis since the eigenvalues of N are symmetric with respect to the real and imaginary axes.

When $\lambda = 0$, the differential equations in (2.1) decouple into two independent equations: $N\mathbf{u} = 0$ if and only if $L_+u = 0$ and $L_-v = 0$. Thus, $\ker(N) = \ker(L_+) \oplus \ker(L_-)$, and $0 \in \mathrm{Spec}(N)$ if and only if $0 \in \mathrm{Spec}(L_+) \cup \mathrm{Spec}(L_-)$. Furthermore, because the eigenvalues of the Sturm–Liouville operators $L_\pm$ are simple,

$$(2.13) \qquad \begin{aligned} \dim\ker(N) = 1 &\iff 0 \in \mathrm{Spec}(L_-) \triangle \mathrm{Spec}(L_+), \\ \dim\ker(N) = 2 &\iff 0 \in \mathrm{Spec}(L_-) \cap \mathrm{Spec}(L_+), \end{aligned}$$

where $A \triangle B := A \cup B \setminus A \cap B$ denotes the symmetric difference. In our application to the stability of standing waves of (1.1), note that (2.9) is equivalent to $L_-\phi = 0$, while autonomy of this equation yields $L_+\phi' = 0$. The boundary conditions satisfied by ϕ therefore influence whether $0 \in \mathrm{Spec}(L_\pm)$. For instance, if ϕ satisfies the Dirichlet conditions (2.10), then $0 \in \mathrm{Spec}(L_-)$ with eigenfunction ϕ, whereas if ϕ satisfies the Neumann conditions (2.11), then $0 \in \mathrm{Spec}(L_+)$ with eigenfunction ϕ', provided ϕ is nonconstant. It is also possible that $0 \notin \mathrm{Spec}(L_+) \cup \mathrm{Spec}(L_-)$ if, for example, more general Robin boundary conditions are imposed on ϕ.

In any of these cases, that L_+ and L_- have nontrivial kernel simultaneously is nongeneric, and so we make this an assumption when studying the stability of NLS standing waves. We stress that the general setup of the paper is given by (2.1)–(2.3), and the following hypothesis is *not* assumed throughout; we will explicitly state whenever we make use of it.

Hypothesis 2.5. N is of the form (2.2)–(2.3), where
 (i) the potentials g and h come from the linearization of the NLS equation (1.1) about a standing wave $\widehat{\psi}$ (and hence are given by (2.12));
 (ii) $0 \notin \mathrm{Spec}(L_-) \cap \mathrm{Spec}(L_+)$.

Remark 2.6. With g and h arbitrary functions of x in general, the results of this paper concerning the stability of NLS standing waves are valid for a spatially dependent nonlinearity $f(x, |\psi|^2)$ as appearing in, for example, [48, 32]. In this case, the loss of autonomy in the standing wave equation (2.9) means that $L_+\phi' \neq 0$; thus, only the results which rely on ϕ' being an eigenfunction for L_+ (Corollary 2.8, Proposition 5.3 and Corollary 5.5) do not generalize to the nonautonomous case.

Under the assumptions of Hypothesis 2.5, our analogue of the Jones–Grillakis instability theorem will follow from both Theorem 2.2 and a computation of the values of $\mathfrak{c}$ given in Theorem 4.14.

COROLLARY 2.7. *Let N be an operator as in (2.2)–(2.3). If $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ and $P - Q \neq -1, 0$ or $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$ and $P - Q \neq 0, 1$, then $n_+(N) \geq 1$. Under Hypothesis 2.5, $\widehat{\psi}$ is spectrally unstable in these cases.*

(The proof is given in subsection 5.1.) This criterion leads to the following instability result. The waves described correspond, for example, to the periodic orbits represented by the phase curves that are contained inside either of the orbits homoclinic to $(0, 0)$ in Figure 1(a).

COROLLARY 2.8. *Assume Hypothesis 2.5. Standing waves satisfying the Neumann boundary conditions (2.11) that are nonconstant and nonvanishing over $[0, \ell]$ and that have one or more critical points in $(0, \ell)$ are unstable.*

(The proof is given in subsection 5.1.) To effectively use Theorem 2.2, we need to understand the quantity $\mathfrak{c}$ appearing in (2.7). Its definition involves the Maslov index at a potentially degenerate crossing and hence requires some work to calculate. We do this by analyzing the curves in the λs-plane that describe the evolution of the real eigenvalues λ of the restricted problem (2.6) as s is varied. As will be seen in Theorem 4.14, $\mathfrak{c}$ is determined by the concavity of these curves. Below, dot denotes $d/d\lambda$. The proof of the following theorem is given in subsection 4.2.

THEOREM 2.9. *Let N be an operator as in (2.2)–(2.3). If $\dim \ker(N) = 1$, then there exists a smooth function $s(\lambda)$, defined for $|\lambda| \ll 1$, such that $s(0) = 1$ and λ is an eigenvalue of (2.6) on $[0, s(\lambda)\ell]$. Moreover, $\dot{s}(0) = 0$, and the concavity of $s(\lambda)$ can be determined as follows:*
   1. *If $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$ with eigenfunction $v \in \ker(L_-)$, then*

$$(2.14) \qquad \ddot{s}(0) = \frac{2}{\ell} \frac{\langle \widehat{u}, v \rangle}{(v'(\ell))^2},$$

   where $\widehat{u} \in H^2(0, \ell) \cap H_0^1(0, \ell)$ is the unique solution to $L_+ \widehat{u} = v$.
   2. *If $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ with eigenfunction $u \in \ker(L_+)$, then*

$$(2.15) \qquad \ddot{s}(0) = -\frac{2}{\ell} \frac{\langle \widehat{v}, u \rangle}{(u'(\ell))^2},$$

   where $\widehat{v} \in H^2(0, \ell) \cap H_0^1(0, \ell)$ is the unique solution to $-L_- \widehat{v} = u$.

Remark 2.10. In applications, we will primarily be interested in the *sign* of $\ddot{s}(0)$, for which (2.14) and (2.15) give

$$(2.16) \qquad \mathrm{sign}\, \ddot{s}(0) = \mathrm{sign} \int_0^\ell \widehat{u} \, v \, dx \qquad \text{and} \qquad \mathrm{sign}\, \ddot{s}(0) = -\mathrm{sign} \int_0^\ell \widehat{v} \, u \, dx,$$

respectively. The integrals in (2.16) can be rewritten as

$$(2.17) \qquad \int_0^\ell \widehat{u} \, v \, dx = \int_0^\ell \widehat{u} \, (L_+ \widehat{u}) \, dx \qquad \text{and} \qquad \int_0^\ell \widehat{v} \, u \, dx = \int_0^\ell \widehat{v} \, (L_- \widehat{v}) \, dx.$$

Consequently, $\ddot{s}(0) > 0$ if $0 \in \mathrm{Spec}(L_-)$ and L_+ is a strictly positive operator or if $0 \in \mathrm{Spec}(L_+)$ and L_- is strictly positive.

In section 4, we will prove a more general version of Theorem 2.9; see Theorem 4.5. An analogous result for the case when $\dim \ker(N) = 2$ is given in Theorem 4.6. Using these results, we give a computation of the Maslov index at the nonregular crossing in Theorem 4.14.

As an application of our theory, working under Hypothesis 2.5, we provide a new formula for the sign of $\ddot{s}(0)$ by evaluating the integral expression in (2.15) for stationary states satisfying (2.11); see Proposition 5.3. In the edge cases when $P - Q = 1$ and $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$ or $P - Q = -1$ and $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$, we show (see Theorem 2.11) that spectral stability of the standing wave $\widehat{\psi}$ is determined by the sign of $\ddot{s}(0)$. This suggests that on a bounded interval, the integrals $\langle \cdot, \cdot \rangle$ in (2.14) and (2.15) play the same role that (1.3) plays in the well-known VK criterion on the real line. We thus refer to the two integral expressions in (2.16) as *VK-type integrals*. In subsection 5.3.2, we show that it is possible to recover the classical VK criterion on a compact interval using the numerator in (2.14) (but *not* (2.15)).

THEOREM 2.11. *Let N be an operator as in* (2.2)–(2.3). *Consider the case when $P = 1$, $Q = 0$, and $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$. If the associated VK-type integral in* (2.14) *is positive, then $n_+(N) = 1$, while if the integral is negative, then $\mathrm{Spec}(N) \subset i\mathbb{R}$. In particular, under Hypothesis 2.5, $\widehat{\psi}$ is spectrally unstable if* (2.14) *is positive and spectrally stable if* (2.14) *is negative.*

Similarly, consider the case when $Q = 1$, $P = 0$, and $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$. If the VK-type integral in (2.15) *is negative, then $n_+(N) = 1$, while if the integral is positive, then $\mathrm{Spec}(N) \subset i\mathbb{R}$. In particular, under Hypothesis 2.5, $\widehat{\psi}$ is spectrally unstable if* (2.15) *is positive and spectrally stable if* (2.15) *is negative.*

(The proof is given in subsection 5.2.) The proofs that $n_+(N) = 1$ rely on an argument that allows the replacement of the inequality in (2.7) with an equality as well as a computation of $\mathfrak{c}$ that yields 1 on the right-hand side of (2.7). The former comes from the fact that the Maslov index is monotone in λ *provided either P or Q is zero* (see Lemma 5.2). On the other hand, to prove $\mathrm{Spec}(N) \subset i\mathbb{R}$ in the cases described in Theorem 2.11, it will be shown (see Lemma 5.1) that the nonnegativity of L_+ or L_- forces the spectrum of N to be confined to the real and imaginary axes. It will then follow from monotonicity in λ (i.e., Lemma 5.2) that $n_+(N) = 0$ (and therefore that $\mathrm{Spec}(N) \subset i\mathbb{R}$).

Remark 2.12. In Theorem 2.11, we recover the equality in [45, Theorem 2.25] without the assumption that the operators $L_\pm$ are invertible (albeit in the case when $P = 0$ or $Q = 0$). Recovering the equality (when L_+ and L_- are invertible) in cases when both P and Q are nonzero via our Maslov index calculations remains an open question.

3. A symplectic approach to the eigenvalue problem. In this section, we review the definition of the Maslov index and give a symplectic formulation of the eigenvalue problem (2.1), culminating in the proof of Theorem 2.2.

3.1. The Maslov index. We begin with some background material on the Maslov index [63]. We follow the definition given by Robbin and Salamon [69], wherein the Maslov index is first defined for regular paths and then extended to arbitrary continuous paths by a homotopy argument. For more on the topological properties of the spaces discussed, see [1]. For a systematic and unified treatement of the Maslov index, featuring an axiomatic description and four equivalent definitions, see [8].

The starting point is $\mathbb{R}^{2n}$ equipped with the nondegenerate, skew-symmetric bilinear form

(3.1) $$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \qquad \omega(x, y) = Jx \cdot y,$$

called a *symplectic form*, where "·" is the dot product in $\mathbb{R}^{2n}$ and J is given in (1.4). A *Lagrangian subspace* or *plane* Λ of $\mathbb{R}^{2n}$ is an n-dimensional subspace on which the

symplectic form vanishes. The *Lagrangian Grassmannian* is the set of all Lagrangian subspaces, $\mathcal{L}(n) = \left\{ \Lambda \subset \mathbb{R}^{2n} : \dim(\Lambda) = n, \quad \omega(x,y) = 0, \, \forall \, x, y \in \Lambda \right\}$. This space has an infinite cyclic fundamental group, i.e., $\pi_1(\mathcal{L}(n)) = \mathbb{Z}$. A notion of winding therefore exists for paths in $\mathcal{L}(n)$; this is the Maslov index. Namely, the Maslov index of a loop in $\mathcal{L}(n)$ is its equivalence class in the fundamental group. Poincaré duality [37, section 3.3] affords an interpretation of this winding number as the (signed) number of intersections with a distinguished codimension-one submanifold, and this allows one to extend the definition to *any* path in $\mathcal{L}(n)$. This is the approach of Arnol'd, which we briefly review.

Fix a reference plane $\Lambda_0 \in \mathcal{L}(n)$. The distinguished codimension-one submanifold of $\mathcal{L}(n)$ is given by the top stratum $\mathcal{T}_1(\Lambda_0)$ of the *train* of Λ_0,

$$\mathcal{T}(\Lambda_0) = \left\{ \Lambda \in \mathcal{L}(n) : \Lambda \cap \Lambda_0 \neq \{0\} \right\} = \bigcup_{k=1}^{n} \mathcal{T}_k(\Lambda_0),$$

where $\mathcal{T}_k(\Lambda_0) = \{\Lambda \in \mathcal{L}(n) : \dim(\Lambda \cap \Lambda_0) = k\}$. As the fundamental lemma of [1] states, $\mathcal{T}_1(\Lambda_0)$ is two-sidedly embedded in $\mathcal{L}(n)$. This means there exists a continuous vector field transverse to $\mathcal{T}_1(\Lambda_0)$ and tangent to $\mathcal{L}(n)$. One can therefore assign a signature to each transverse intersection of a path in $\mathcal{L}(n)$ with $\mathcal{T}_1(\Lambda_0)$. Any Lagrangian path with endpoints not in $\mathcal{T}(\Lambda_0)$ can be perturbed to one that only intersects $\mathcal{T}_1(\Lambda_0)$ and only does so transversally; the Maslov index is then defined to be the sum of the signatures of all such intersections.

We next recall the approach of Robbin and Salamon [69], which requires additional regularity but applies to paths whose endpoints are in the train and also allows for intersections with $\mathcal{T}_k(\Lambda_0)$ when $k \geq 2$. This approach, while less geometric than the above interpretation of the Maslov index as an intersection number, is more suited to practical computations.

Given a smooth path $\Lambda : [a,b] \longrightarrow \mathcal{L}(n)$, a *crossing* is a point $t = t_0$, where $\Lambda(t_0) \in \mathcal{T}(\Lambda_0)$. Let $\Lambda_0^\perp \subset \mathbb{R}^{2n}$ be a subspace transverse to $\Lambda(t_0)$. Then $\Lambda_0^\perp$ is transverse to $\Lambda(t)$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ for ε small enough. Thus, there exists a smooth family of matrices $R_t : \Lambda(t_0) \to \Lambda_0^\perp$ such that

$$(3.2) \qquad\qquad \Lambda(t) = \operatorname{graph}(R_t) = \{q + R_t q : q \in \Lambda(t_0)\}$$

for $|t - t_0| \leq \varepsilon$, where $R_{t_0}|_{\Lambda(t_0)} \equiv 0$. At a crossing t_0, the *crossing form* is the quadratic form

$$(3.3) \qquad \mathfrak{m}_{t_0}(q) = \frac{d}{dt}\omega(q, q + R_t q)\Big|_{t=t_0} = \omega(q, \dot{R}_{t_0} q), \qquad q \in \Lambda(t_0) \cap \Lambda_0,$$

on the intersection $\Lambda(t_0) \cap \Lambda_0$. The full symmetric bilinear form associated with the quadratic form (3.3) may be recovered using the polarisation identity; see, for example, the proof of Corollary 3.10. A crossing is called *regular* if the form (3.3) is nondegenerate and *simple* if $\Lambda(t_0) \in \mathcal{T}_1(\Lambda_0)$. Since $\mathfrak{m}_{t_0}$ is quadratic, it may be diagonalised; we let $n_+(\mathfrak{m}_{t_0})$ and $n_-(\mathfrak{m}_{t_0})$ be the number of positive and negative squares obtained in so doing. The signature of $\mathfrak{m}_{t_0}$ is the integer sign $(\mathfrak{m}_{t_0}) = n_+(\mathfrak{m}_{t_0}) - n_-(\mathfrak{m}_{t_0})$. We then define the Maslov index as follows.

DEFINITION 3.1. *The Maslov index for a path* $\Lambda : [a,b] \longrightarrow \mathcal{L}(n)$ *having only regular crossings is given by*

$$(3.4) \qquad \operatorname{Mas}(\Lambda(t), \Lambda_0; [a,b]) := -n_-(\mathfrak{m}_a) + \sum_{a < t_0 < b} \operatorname{sign}(\mathfrak{m}_{t_0}) + n_+(\mathfrak{m}_b),$$

where the sum is taken over all crossings $t_0 \in (a,b)$.

One can show that regular crossings are isolated, and therefore the sum is well-defined. Note the convention at the endpoints: At $t = a$, only the negative squares contribute to the Maslov index, while at $t = b$, only the positive squares contribute. Other conventions are possible (see, e.g., [69, section 2]), but we choose the above in order to ensure the Maslov index is an integer.

The Maslov index of an arbitrary continuous path $\Lambda_1 : [a, b] \longrightarrow \mathcal{L}(n)$ is then defined to be $\mathrm{Mas}(\Lambda_2(t), \Lambda_0; [a, b])$, where Λ_2 is any path that is homotopic (with fixed endpoints) to Λ_1 and has only regular crossings. It is guaranteed by [69, Lemmas 2.1 and 2.2] that such a path exists, and any two such paths have the same index, so the Maslov index of Λ_1 is well-defined.

The essential properties of the Maslov index that we will use are given in the following proposition; see [69, Theorem 2.3].

PROPOSITION 3.2. *The Maslov index enjoys the following:*
1. *Homotopy invariance: If two paths $\Lambda_1, \Lambda_2 : [a, b] \longrightarrow \mathcal{L}(n)$ are homotopic with fixed endpoints, then*

$$(3.5) \qquad \mathrm{Mas}(\Lambda_1(t), \Lambda_0; [a, b]) = \mathrm{Mas}(\Lambda_2(t), \Lambda_0; [a, b]).$$

2. *Additivity under concatenation: For $\Lambda(t) : [a, c] \longrightarrow \mathcal{L}(n)$ and $a < b < c$,*

$$(3.6) \qquad \mathrm{Mas}(\Lambda(t), \Lambda_0; [a, c]) = \mathrm{Mas}(\Lambda(t), \Lambda_0; [a, b]) + \mathrm{Mas}(\Lambda(t), \Lambda_0; [b, c]).$$

To conclude our discussion of the Maslov index, we expound the notion of a nonregular crossing, that is, a crossing with degenerate crossing form. Consider a Lagrangian path $\Lambda : [a, b] \longrightarrow \mathcal{L}(n)$ with a nonregular crossing $t = t_0$. In the case that $\mathfrak{m}_{t_0}$ is identically zero, in [23, Proposition 3.10], the authors state that the contribution to the Maslov index is determined by the second-order crossing form

$$(3.7) \qquad \mathfrak{m}_{t_0}^{(2)}(q) := \frac{d^2}{dt^2}\omega(q, q + R_t q)\Big|_{t=t_0} = \omega(q, \ddot{R}_{t_0} q), \qquad q \in \Lambda(t_0) \cap \Lambda_0,$$

provided it is nondegenerate. Such a crossing can only contribute to the Maslov index if it occurs at one of the endpoints: If $t_0 = a$, then it contributes $-n_-(\mathfrak{m}_a^{(2)})$, and if $t_0 = b$, then it contributes $n_+(\mathfrak{m}_b^{(2)})$.

As an example, consider the case of a simple crossing with $\mathfrak{m}_{t_0} = 0$ but $\mathfrak{m}_{t_0}^{(2)} \neq 0$. In the Lagrangian Grassmannian, this corresponds to our path Λ tangentially intersecting the train $\mathcal{T}(\Lambda_0)$ of the fixed reference plane to quadratic order; i.e., Λ "bounces off" the train as t passes through t_0. Provided t_0 lies in the interior of $[a, b]$, the contribution to the Maslov index will be zero: Clearly, the path can locally be homotoped to one with no crossings at all. If $t_0 = a$, the contribution is -1 provided the path leaves in the negative direction (and zero otherwise), while if $t_0 = b$, the contribution is $+1$ provided the path arrives in the positive direction (and zero otherwise). If the second-order form is degenerate, i.e., $\mathfrak{m}_{t_0}^{(2)} = 0$, higher-order derivatives are needed in order to determine the local behavior of the path Λ.

In the present setting, with the spectral parameter λ acting as the independent variable, we will observe that a nonregular crossing occurs at $\lambda = 0$. To determine the contribution to the Maslov index of this nonregular crossing, we use a homotopy argument, made possible by our analysis of the local behavior of the eigenvalue curves in subsection 4.4. We confirm that our computation agrees with the number of negative squares of the second-order form (3.7) used in [23]. For a further discussion of nonregular crossings and an alternate way to compute the Maslov index at such points, see [29, 30].

3.2. Spatial rescaling and construction of the Lagrangian path. We now view the problem through the lens of the Lagrangian formalism by interpreting eigenvalues as nontrivial intersections of Lagrangian planes. Following the approach of [23], we restrict the eigenvalue problem to a family of subintervals $[0, s\ell]$ for $s \in (0,1)$. Rescaling the equations to the full domain $[0, \ell]$, we construct a two-parameter family of Lagrangian subspaces in s and λ via rescaled boundary traces of solutions to the system of differential equations without any boundary conditions at all. An eigenvalue is produced when this family of subspaces nontrivially intersects a fixed reference plane that encodes Dirichlet boundary conditions. Identifying a Lagrangian structure boils down to a judicious choice of both the symplectic form and the definition of the trace map: if we employ the standard symplectic form ω in (3.1), then we need to carefully define the trace map (3.10) such that the space of boundary traces is Lagrangian with respect to ω. We begin by introducing some notation.

We let

$$(3.8) \qquad N = D + B(x), \qquad D := \begin{pmatrix} 0 & \partial_{xx} \\ -\partial_{xx} & 0 \end{pmatrix}, \qquad B(x) := \begin{pmatrix} 0 & h(x) \\ -g(x) & 0 \end{pmatrix}$$

and introduce the s-dependent operators acting on functions on $[0, \ell]$,

$$(3.9) \quad B_s(x) := s^2 B(sx), \qquad N_s := \begin{pmatrix} 0 & -L_-^s \\ L_+^s & 0 \end{pmatrix}, \qquad \begin{cases} L_+^s := -\partial_{xx} - s^2 g(sx) \\ L_-^s := -\partial_{xx} - s^2 h(sx) \end{cases}$$

so that $N_s = D + B_s(x)$. We define the *rescaled trace* of $\mathbf{u} = (u,v)^\top \in H^2(0,\ell)$ as the vector

$$(3.10) \quad \mathrm{Tr}_s \mathbf{u} := \left(u(0), v(0), u(\ell), v(\ell), -\frac{1}{s} u'(0), \frac{1}{s} v'(0), \frac{1}{s} u'(\ell), -\frac{1}{s} v'(\ell) \right)^\top \in \mathbb{R}^8$$

and denote the vertical subspace of $\mathbb{R}^8$ by $\mathcal{D} := \{0\} \times \mathbb{R}^4$. Using the above notation, we may rewrite the restricted problem (2.6) as a boundary value problem on $[0, \ell]$. Indeed, if $\mathbf{u}(x) \in H^2(0, s\ell) \cap H_0^1(0, s\ell)$, then $\mathbf{u}_s(x) := \mathbf{u}(sx) \in H^2(0, \ell) \cap H_0^1(0, \ell)$. It follows from (3.10) that $\mathbf{u}(0) = \mathbf{u}(s\ell) = 0$ if and only if $\mathrm{Tr}_s \mathbf{u}_s \in \mathcal{D}$. Thus, rescaled to $[0, \ell]$, (2.6) reads

$$(3.11) \qquad\qquad\qquad N_s \mathbf{u}_s = s^2 \lambda \mathbf{u}_s, \quad \mathrm{Tr}_s \mathbf{u}_s \in \mathcal{D}.$$

Note that the solution spaces of the boundary value problems (2.6) and (3.11) are isomorphic: $\mathbf{u} = (u,v)^\top \in \mathrm{dom}(N|_{[0,s\ell]})$ solves (2.6) if and only if $\mathbf{u}_s = (u_s, v_s)^\top \in \mathrm{dom}(N_s)$ solves (3.11). Consequently, λ is an eigenvalue of $N|_{[0,s\ell]}$ if and only if $s^2 \lambda$ is an eigenvalue of N_s.

Remark 3.3. The rescaled problem (3.11) is well-defined for $s > 1$ provided the potentials g and h are defined for $x > \ell$. In this case, the "restricted" eigenvalue problem (2.6) corresponds to a *stretching* of the domain.

Remark 3.4. As per Remark 2.1, notationally we will not distinguish between N_s and $L_\pm^s$ as differential expressions and as unbounded operators with dense domains given by (2.4) and (2.5), respectively. Thus, when we write $s^2 \lambda \in \mathrm{Spec}(N_s)$ or $\mathbf{u}_s \in \ker(N_s - s^2 \lambda)$, we mean that (3.11) is solved for some eigenfunction $\mathbf{u}_s$; similar statements hold when $\lambda \in \mathrm{Spec}(L_\pm^s)$.

That the formulation (3.11) lends itself to a symplectic interpretation can be seen via the following modified version of Green's second identity. Using our definition of

the rescaled trace map (3.10) and the symplectic form (3.1), one can verify that for each $s \in (0,1]$ and all $\mathbf{u}, \mathbf{v} \in H^2(0,\ell)$,

$$(3.12) \qquad \langle S(N_s - s^2\lambda)\mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, S(N_s - s^2\lambda)\mathbf{v}\rangle = s\omega(\mathrm{Tr}_s\mathbf{u}, \mathrm{Tr}_s\mathbf{v}),$$

where S is defined in (1.4). Now define the space

$$(3.13) \qquad \mathcal{K}_{\lambda,s} := \left\{\mathbf{u} \in H^2(0,\ell) : (N_s - s^2\lambda)\mathbf{u} = 0 \text{ in } L^2(0,\ell)\right\}$$

of all solutions to the homogeneous differential equation $N_s\mathbf{u} = s^2\lambda\mathbf{u}$ *without* any reference to the boundary conditions, so that $\ker(N_s - s^2\lambda) = \mathcal{K}_{\lambda,s} \cap H_0^1(0,\ell)$.

Remark 3.5. The trace map is an injective linear operator on the space $\mathcal{K}_{\lambda,s}$. If $\mathbf{u}_s \in \mathcal{K}_{\lambda_0,s}$, then $\mathrm{Tr}_s\mathbf{u}_s = 0$ implies $\mathbf{u}_s = 0$ since $\mathbf{u}_s$ solves a system of second-order equations.

Taking the (rescaled) boundary trace leads to the desired family of Lagrangian subspaces with respect to the form ω in (3.1).

LEMMA 3.6. *The space*

$$(3.14) \qquad \Lambda(\lambda, s) := \mathrm{Tr}_s(\mathcal{K}_{\lambda,s}) = \{\mathrm{Tr}_s(\mathbf{u}) : \mathbf{u} \in \mathcal{K}_{\lambda,s}\}$$

is a Lagrangian subspace of $\mathbb{R}^8$ for all $s \in (0,1]$ and all $\lambda \in \mathbb{R}$.

Proof. Fix $\lambda \in \mathbb{R}$ and $s \in (0,1]$. From (3.12), for $\mathbf{u}, \mathbf{v} \in \mathcal{K}_{\lambda,s}$, we have $\omega(\mathrm{Tr}_s\mathbf{u}, \mathrm{Tr}_s\mathbf{v}) = 0$. Since $\mathcal{K}_{\lambda,s}$ is the space of solutions to a system of two second-order differential equations, $\dim \mathcal{K}_{\lambda,s} = 4$. Hence, $\dim \mathrm{Tr}_s(\mathcal{K}_{\lambda,s}) = 4$, and $\mathrm{Tr}_s(\mathcal{K}_{\lambda,s}) \in \mathcal{L}(4)$ is Lagrangian. $\qquad\square$

We now have the desired interpretation of eigenvalues as nontrivial intersections of Lagrangian subspaces.

PROPOSITION 3.7. $s^2\lambda \in \mathrm{Spec}(N_s)$ *if and only if* $\Lambda(\lambda, s) \cap \mathcal{D} \neq \{0\}$. *Moreover, the geometric multiplicity of the eigenvalue is equal to the dimension of the Lagrangian intersection:*

$$(3.15) \qquad \dim \ker(N_s - s^2\lambda) = \dim \Lambda(\lambda, s) \cap \mathcal{D}.$$

Proof. The first statement follows from the definition of Λ. Equality (3.15) follows from the injectivity (and thus bijectivity) of the trace map acting between the finite-dimensional spaces $\ker(N_{s_0} - s_0^2\lambda_0) = \mathcal{K}_{\lambda_0,s_0} \cap H_0^1(0,\ell)$ and $\mathrm{Tr}_{s_0}(\mathcal{K}_{\lambda_0,s_0} \cap H_0^1(0,\ell)) = \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. $\qquad\square$

Hereafter, a *crossing* refers to a pair $(\lambda, s) = (\lambda_0, s_0)$ such that $\Lambda(\lambda_0, s_0) \cap \mathcal{D} \neq \{0\}$, while a *conjugate point* refers to a crossing for which $\lambda_0 = 0$. It follows from Proposition 3.7 that crossings where $s_0 = 1$ correspond to eigenvalues of the operator N on $[0,\ell]$.

To prove Theorem 2.2, our goal then is to bound from below the number of crossings for which $s_0 = 1, \lambda_0 > 0$. To do so, we use a homotopy argument that involves appropriately counting conjugate points. In order to set this argument up, we introduce in Figure 2 the so-called *Maslov box,* given by the boundary Γ of the rectangle $[0, \lambda_\infty] \times [\tau, 1]$ in the λs-plane, where $\tau > 0$ is small and $\lambda_\infty > 0$ is large.

Since $\Lambda : [0, \lambda_\infty] \times [\tau, 1] \longrightarrow \mathcal{L}(4)$ is a continuous map, the image $\Lambda(\Gamma)$ of the Maslov box is null homotopic, and so

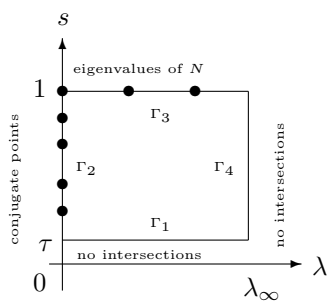$$(3.16) \qquad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma) = 0.$$

FIG. 2. *Maslov box in the λs-plane.*

We partition Γ into its constituent sides such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$
(3.17) \quad
\begin{aligned}
&\Gamma_1 : s = \tau, \quad 0 \le \lambda \le \lambda_\infty, \qquad &&\Gamma_3 : s = 1, \quad 0 \le \lambda \le \lambda_\infty, \\
&\Gamma_2 : \lambda = 0, \quad \tau \le s \le 1, \qquad &&\Gamma_4 : \lambda = \lambda_\infty, \quad \tau \le s \le 1,
\end{aligned}
$$

(see Figure 2) and assign a direction to each of these intervals such that the entirety of the Maslov box is oriented in a clockwise fashion. We then appeal to the concatenation property in Proposition 3.2 to rewrite (3.16) as

$$(3.18) \quad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_1) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0.$$

Taking $\lambda = \lambda_\infty$ large enough and $s = \tau$ small enough, it will follow (see Lemma 3.23) that there are no crossings along Γ_1 and Γ_4 and therefore that the Maslov indices of these pieces are zero. The crossing forms needed to analyze $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2)$ and $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3)$ are given in the next section.

3.3. Crossing forms. Our next task is the calculation of the crossing forms (3.3) associated with the trajectories through the crossing (λ_0, s_0), where $\lambda = \lambda_0$ is held constant and s increases and vice versa. The key ingredient will be the Green's-type identity (3.12). The approach is inspired by Lemma 4.18 and the proof of Theorem 4.19 in [58] as well as the crossing form calculation in [18, Lemma 5.2]. Before proceeding, we set some notation that will be useful in this section and throughout the rest of the paper.

Remark 3.8. We denote by $\mathbf{u}_{s_0}$ any eigenfunction $\mathbf{u}_{s_0} \in \ker(N_{s_0} - s_0^2 \lambda_0)$, and when $s_0 = 1$, we drop the subscript. If $\dim \ker(N_{s_0} - s_0^2 \lambda_0) = n$, we denote a basis for this space by $\{\mathbf{u}_{s_0}^{(1)}, \ldots, \mathbf{u}_{s_0}^{(n)}\}$, where $\mathbf{u}_{s_0}^{(i)} = \left(u_{s_0}^{(i)}, v_{s_0}^{(i)}\right)^\top$. The set $\{S\mathbf{u}_{s_0}^{(1)}, \ldots, S\mathbf{u}_{s_0}^{(n)}\}$ is then a basis for the kernel of the adjoint operator, $\ker(N_{s_0}^* - s_0^2 \lambda_0)$, since λ_0 is real. Note that S (given in (1.4)) merely swaps the entries of the vector it acts on. When $s_0 = 1$, we denote

$$(3.19) \quad \mathbf{u}_i := \mathbf{u}_1^{(i)}, \qquad u_i := u_1^{(i)}, \ v_i := v_1^{(i)}.$$

Because $\ker(N_{s_0}) = \ker(L_+^{s_0}) \oplus \ker(L_-^{s_0})$, when $\lambda_0 = 0$ and $\dim \ker(N_{s_0}) = 1$, we have

$$(3.20) \quad \mathbf{u}_{s_0} = \begin{cases} (u_{s_0}, 0)^\top, & 0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0}), \quad \ker(L_+^{s_0}) = \mathrm{Span}\{u_{s_0}\}, \\ (0, v_{s_0})^\top, & 0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0}), \quad \ker(L_-^{s_0}) = \mathrm{Span}\{v_{s_0}\}. \end{cases}$$

When $\lambda_0 = 0$ and $\dim \ker(N_{s_0}) = 2$, we denote

$$(3.21) \quad \mathbf{u}_{s_0}^{(1)} = \begin{pmatrix} u_{s_0}^{(1)} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_{s_0}^{(2)} = \begin{pmatrix} 0 \\ v_{s_0}^{(2)} \end{pmatrix},$$

where $\ker(L_+^{s_0}) = \mathrm{Span}\{u_{s_0}^{(1)}\}$ and $\ker(L_-^{s_0}) = \mathrm{Span}\{v_{s_0}^{(2)}\}$.

In the current paper, where the potentials g and h from (2.3) are scalar-valued, we will always have $n \leq 2$. However, if g and h are matrix-valued (and symmetric), so that $L_\pm$ are systems of self-adjoint Schrödinger operators, or if the operator N acts on functions on a multidimensional domain, then we may have $n > 2$. The results in this section and section 4 have been stated for a general n to indicate how the theory extends to these cases.

Returning to our computation of crossing forms, we first compute the crossing form (3.3) for the path of Lagrangian planes $s \mapsto \Lambda(\lambda_0, s)$, holding $\lambda = \lambda_0$ fixed. Recall that $N_s = D + B_s$, as in (3.9), and that $S = S^T$.

LEMMA 3.9. *Let (λ_0, s_0) be a crossing, and fix any nonzero $q \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. Then there exists a unique $\mathbf{u}_{s_0} \in \mathcal{K}_{\lambda_0, s_0}$ such that $q = \mathrm{Tr}_{s_0} \mathbf{u}_{s_0}$, and the crossing form for the Lagrangian path $s \mapsto \Lambda(\lambda_0, s)$ at $s = s_0$ is given by*

$$(3.22) \qquad \mathfrak{m}_{s_0}(q) = \frac{1}{s_0} \left\langle \left(\partial_s B_{s_0} - 2s_0\lambda_0\right)\mathbf{u}_{s_0}, S\mathbf{u}_{s_0} \right\rangle,$$

where $\partial_s B_s = 2sB(sx) + s^2 B'(sx)x$. In particular, along Γ_2, where $\lambda_0 = 0$, we have

$$(3.23) \qquad \mathfrak{m}_{s_0}(q) = \frac{\ell}{s_0^2} \left[-\left(u'_{s_0}(\ell)\right)^2 + \left(v'_{s_0}(\ell)\right)^2 \right].$$

In this case, if the crossing $(0, s_0)$ is simple, then the form (3.23) is nondegenerate.

Proof. Consider a C^1 family of vectors $s \mapsto \mathbf{w}_s \in \mathcal{K}_{\lambda_0, s}$ satisfying

$$(3.24a) \qquad N_s \mathbf{w}_s = s^2 \lambda_0 \mathbf{w}_s, \qquad x \in [0, \ell], \quad s \in (s_0 - \varepsilon, s_0 + \varepsilon),$$

$$(3.24b) \qquad \mathrm{Tr}_s \mathbf{w}_s = \mathrm{Tr}_{s_0} \mathbf{u}_{s_0} + R_s \mathrm{Tr}_{s_0} \mathbf{u}_{s_0}, \qquad \mathbf{w}_{s_0} = \mathbf{u}_{s_0},$$

where $R_s : \Lambda(\lambda_0, s_0) \to \mathcal{D}^\perp$ is the smooth family of matrices such that $\Lambda(\lambda_0, s) = \mathrm{graph}(R_s)$; cf.(3.2). To prove the existence of such a family $s \mapsto \mathbf{w}_s$, consider the smooth family of vectors $h_s := q + R_s q \in \Lambda(\lambda_0, s)$, where $h_{s_0} = q$ since $R_{s_0} q = 0$ for all $q \in \Lambda(\lambda_0, s_0)$. The injectivity (and thus bijectivity) of the linear map

$$\mathrm{Tr}_s : \mathcal{K}_{\lambda_0, s} \longrightarrow \mathrm{Tr}_s(\mathcal{K}_{\lambda_0, s}) = \Lambda(\lambda_0, s)$$

(see Remark 3.5) then implies that for each $h_s \in \Lambda(\lambda_0, s)$, there exists a unique $\mathbf{w}_s \in \mathcal{K}_{\lambda_0, s}$ such that $\mathrm{Tr}_s \mathbf{w}_s = h_s$ and in particular $\mathrm{Tr}_{s_0} \mathbf{w}_{s_0} = h_{s_0} = q$.

We now turn to the computation of (3.3). We have

$$\begin{aligned} \mathfrak{m}_{s_0}(q) &= \frac{d}{ds}\,\omega(q, R_s q)\bigg|_{s=s_0} \\ &= \frac{d}{ds}\,\omega(\mathrm{Tr}_{s_0}\mathbf{u}_{s_0}, \mathrm{Tr}_s \mathbf{w}_s)\bigg|_{s=s_0} \\ &= \omega\left(\mathrm{Tr}_{s_0}\mathbf{u}_{s_0}, \frac{d}{ds}\mathrm{Tr}_s\big|_{s=s_0}\mathbf{u}_{s_0}\right) + \omega\left(\mathrm{Tr}_{s_0}\mathbf{u}_{s_0}, \mathrm{Tr}_{s_0}\frac{d}{ds}\mathbf{w}_s\big|_{s=s_0}\right). \end{aligned}$$

The first term is zero since $\mathrm{Tr}_{s_0}\mathbf{u}_{s_0} \in \mathcal{D}$ implies $\mathrm{Tr}_{s_0}\mathbf{u}_{s_0} = \left(0, s_0^{-1}\gamma_N \mathbf{u}_{s_0}\right)$ and $\frac{d}{ds}\mathrm{Tr}_s\big|_{s=s_0}\mathbf{u}_{s_0} = \left(0, -s_0^{-2}\gamma_N \mathbf{u}_{s_0}\right)$, where $\gamma_N \mathbf{u} := \left(-u'(0), v'(0), u'(\ell), -v'(\ell)\right)^\top$. For the second term, we differentiate the equation in (3.24a) with respect to s and apply $\langle \cdot, S\mathbf{w}_s\rangle$:

$$(3.25) \qquad \langle(\partial_s B_s - 2s\lambda_0)\mathbf{w}_s, S\mathbf{w}_s\rangle + \langle(N_s - s^2\lambda_0)\,\partial_s \mathbf{w}_s, S\mathbf{w}_s\rangle = 0.$$

From the Green's-type identity (3.12) with $\mathbf{u} = \mathbf{w}_s$ and $\mathbf{v} = \partial_s \mathbf{w}_s$, we have

$$s\,\omega(\mathrm{Tr}_s\mathbf{w}_s, \mathrm{Tr}_s\partial_s\mathbf{w}_s) = \langle(N_s - s^2\lambda_0)\mathbf{w}_s, S\,\partial_s\mathbf{w}_s\rangle - \langle S\mathbf{w}_s, (N_s - s^2\lambda_0)\partial_s\mathbf{w}_s\rangle,$$

and using (3.24a) and (3.25), this reduces to

$$(3.26) \qquad s\,\omega(\mathrm{Tr}_s\mathbf{w}_s, \mathrm{Tr}_s\partial_s\mathbf{w}_s) = \langle(\partial_s B_s - 2s\lambda_0)\mathbf{w}_s, S\mathbf{w}_s\rangle.$$

Evaluating (3.26) at $s = s_0$ and dividing by s_0, (3.22) follows. When $\lambda_0 = 0$, substituting the stated expression for $\partial_s B_{s_0}$ in (3.22) gives

$$\mathfrak{m}_{s_0}(q) = \langle(2B(s_0 x) + s_0 B'(s_0 x)x)\,\mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle$$
$$= \int_0^\ell \Big\{ [2h(s_0 x) + s_0 x h'(s_0 x)]\,v_{s_0}^2(x) - [2g(s_0 x) + s_0 x g'(s_0 x)]\,u_{s_0}^2(x) \Big\} dx.$$

A direct calculation using the equation $L_-^{s_0} v_{s_0} = 0$, i.e., $v_{s_0}''(x) + s_0^2 h(s_0 x)v_{s_0}(x) = 0$, gives

$$\frac{d}{dx}\left[\frac{1}{s_0^2}x\left(v_{s_0}'(x)\right)^2 + x v_{s_0}^2(x)h(s_0 x) - \frac{1}{s_0^2}v_{s_0}(x)v_{s_0}'(x)\right] = [2h(s_0 x) + s_0 x h'(s_0 x)]\,v_{s_0}^2(x).$$

Integrating and using the fact that $v_{s_0}(0) = v_{s_0}(\ell) = 0$, we get

$$\int_0^\ell [2h(s_0 x) + s_0 x h'(s_0 x)]\,v_{s_0}^2(x)dx = \frac{\ell}{s_0^2}\left(v_{s_0}'(\ell)\right)^2.$$

Computing similarly for the second term, we arrive at (3.23). That the form is nondegenerate in the simple case follows from (3.20): If $\dim\ker(N_{s_0}) = 1$, then exactly one of the entries of $\mathbf{u}_s = (u_s, v_s)^\top \in \ker(N_{s_0})$ is nontrivial. Since this function satisfies a second-order differential equation with Dirichlet boundary conditions, its derivative is nonzero at $x = \ell$, and therefore (3.23) is nonzero. □

COROLLARY 3.10. *Assume* $\dim\ker(N_{s_0} - s_0^2\lambda_0) = n$, *and let* $\{\mathbf{u}_{s_0}^{(1)}, \mathbf{u}_{s_0}^{(2)}, \ldots, \mathbf{u}_{s_0}^{(n)}\}$ *be a basis for* $\ker(N_{s_0} - s_0^2\lambda_0)$. *The* $n \times n$ *symmetric matrix* $\mathfrak{M}_{s_0}$ *induced from the quadratic form* (3.22) *is given by*

$$(3.27) \qquad [\mathfrak{M}_{s_0}]_{ij} = \frac{1}{s_0}\left\langle(\partial_s B_{s_0} - 2s_0\lambda_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)}\right\rangle, \qquad i,j = 1,\ldots,n.$$

Consequently, when $\lambda_0 = 0$ *and* $n = 2$, *the form* $\mathfrak{m}_{s_0}$ *is nondegenerate.*

Proof. Letting $q_i := \mathrm{Tr}_{s_0}\mathbf{u}_{s_0}^{(i)}$, it follows from the linearity and injectivity of the trace map that $\{q_i\}_{i=1}^n$ is a basis for $\Lambda(\lambda_0, s_0) \cap \mathcal{D}$. To construct the symmetric bilinear form associated with the quadratic form (3.22), we compute the off-diagonal terms $\mathfrak{m}_{s_0}(q_i, q_j)$ via the real polarisation identity

$$(3.28) \qquad \mathfrak{m}_{s_0}(q_i, q_j) = \frac{1}{4}\big[\mathfrak{m}_{s_0}(q_i + q_j) - \mathfrak{m}_{s_0}(q_i - q_j)\big].$$

Since both S and $S(\partial_s B_{s_0})$ are symmetric, we obtain

$$\mathfrak{m}_{s_0}(q_i, q_j) = \frac{1}{4}\left\langle(\partial_s B_{s_0} - 2s_0\lambda_0)(\mathbf{u}_{s_0}^{(i)} + \mathbf{u}_{s_0}^{(j)}), S(\mathbf{u}_{s_0}^{(i)} + \mathbf{u}_{s_0}^{(j)})\right\rangle$$
$$- \frac{1}{4}\left\langle(\partial_s B_{s_0} - 2s_0\lambda_0)(\mathbf{u}_{s_0}^{(i)} - \mathbf{u}_{s_0}^{(j)}), S(\mathbf{u}_{s_0}^{(i)} - \mathbf{u}_{s_0}^{(j)})\right\rangle$$
$$= \left\langle(\partial_s B_{s_0} - 2s_0\lambda_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)}\right\rangle.$$

The corresponding matrix elements with respect to the basis $\{q_i\}$ are $[\mathfrak{M}_{s_0}]_{ij} = \mathfrak{m}_{s_0}(q_i, q_j)$, and the first statement of the corollary follows. In the case $\lambda_0 = 0$ and $n = 2$, using (3.23) and recalling (3.21), the matrix (3.27) reduces to

$$(3.29) \qquad \mathfrak{M}_{s_0} = \frac{\ell}{s_0^2} \begin{pmatrix} -\big(\partial_x u_{s_0}^{(1)}(\ell)\big)^2 & 0 \\ 0 & \big(\partial_x v_{s_0}^{(2)}(\ell)\big)^2 \end{pmatrix},$$

which clearly has full rank. Nondegeneracy of the quadratic form $\mathfrak{m}_{s_0}$ follows. □

We now move to the λ-direction. Holding $s = s_0$ fixed, we compute the crossing form (3.3) with respect to λ. We denote $d/d\lambda$ with a dot.

LEMMA 3.11. *Let (λ_0, s_0) be a crossing, and fix any nonzero $q \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. Then there exists a unique $\mathbf{u}_{s_0} \in \mathcal{K}_{\lambda_0, s_0}$ such that $q = \mathrm{Tr}_{s_0} \mathbf{u}_{s_0}$, and the crossing form for the Lagrangian path $\lambda \mapsto \Lambda(\lambda, s_0)$ at $\lambda = \lambda_0$ is given by*

$$(3.30) \qquad \mathfrak{m}_{\lambda_0}(q) = -s_0 \langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0} \rangle = -2s_0 \langle u_{s_0}, v_{s_0} \rangle.$$

Proof. The argument is virtually identical to that in the s direction. Fixing $s = s_0$, we consider a C^1 family of vectors $\lambda \mapsto \mathbf{w}_\lambda \in \mathcal{K}_{\lambda, s_0}$ satisfying

$$(3.31a) \qquad N_{s_0} \mathbf{w}_\lambda = s_0^2 \lambda \mathbf{w}_\lambda, \qquad x \in [0, \ell], \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon),$$

$$(3.31b) \qquad \mathrm{Tr}_{s_0} \mathbf{w}_\lambda = \mathrm{Tr}_{s_0} \mathbf{u}_{s_0} + R_\lambda \mathrm{Tr}_{s_0} \mathbf{u}_{s_0}, \qquad \mathbf{w}_{\lambda_0} = \mathbf{u}_{s_0},$$

where now $R_\lambda : \Lambda(\lambda_0, s_0) \longrightarrow \mathcal{D}^\perp$ is such that $\Lambda(\lambda, s_0) = \mathrm{graph}(R_\lambda)$. Similar to (3.25), we have

$$\langle -s_0^2 \mathbf{w}_\lambda, S\mathbf{w}_\lambda \rangle + \langle (N_{s_0} - s_0^2 \lambda)\dot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle = 0,$$

and using the identity (3.12) with $\mathbf{u} = \mathbf{w}_\lambda$ and $\mathbf{v} = \dot{\mathbf{w}}_\lambda$ yields

$$s_0 \,\omega(\mathrm{Tr}_{s_0} \mathbf{w}_\lambda, \mathrm{Tr}_{s_0} \dot{\mathbf{w}}_\lambda) = \langle (N_{s_0} - s_0^2 \lambda)\mathbf{w}_\lambda, S\dot{\mathbf{w}}_\lambda \rangle - \langle S\mathbf{w}_\lambda, (N_{s_0} - s_0^2 \lambda)\dot{\mathbf{w}}_\lambda \rangle.$$

The previous two equations along with (3.31a) give

$$(3.32) \qquad s_0 \,\omega(\mathrm{Tr}_{s_0} \mathbf{w}_\lambda, \mathrm{Tr}_{s_0} \dot{\mathbf{w}}_\lambda) = -\langle s_0^2 \mathbf{w}_\lambda, S\mathbf{w}_\lambda \rangle.$$

Therefore, the crossing form (3.3) is

$$\mathfrak{m}_{\lambda_0}(q) = \omega\left(\mathrm{Tr}_{s_0} \mathbf{u}_{s_0}, \mathrm{Tr}_{s_0} \dot{\mathbf{w}}_\lambda\big|_{\lambda = \lambda_0}\right) = -s_0 \langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0} \rangle = -2s_0 \langle u_{s_0}, v_{s_0} \rangle,$$

where we used (3.32) evaluated at $\lambda = \lambda_0$. □

Recalling (3.20), at a simple crossing $(0, s_0)$, one of u_{s_0} or v_{s_0} is always trivial. Degeneracy of the λ-crossing form immediately follows.

COROLLARY 3.12. *All conjugate points $(0, s_0)$ for which $\dim \ker(N_{s_0}) = 1$ are nonregular in the λ direction; i.e., at all such points, $\mathfrak{m}_{\lambda_0} = 0$.*

For the case of higher-dimensional crossings, we have the following corollary to Lemma 3.11.

COROLLARY 3.13. *Assume $\dim \ker(N_{s_0} - s_0^2 \lambda_0) = n$, and let $\{\mathbf{u}_{s_0}^{(1)}, \mathbf{u}_{s_0}^{(2)}, \ldots, \mathbf{u}_{s_0}^{(n)}\}$ be a basis for $\ker(N_{s_0} - s_0^2 \lambda_0)$. The $n \times n$ symmetric matrix $\mathfrak{M}_{\lambda_0}$ induced from the n-dimensional quadratic form (3.30) is given by*

$$(3.33) \qquad [\mathfrak{M}_{\lambda_0}]_{ij} = -s_0 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \qquad i, j = 1, \ldots n.$$

Consequently, when $\lambda_0 = 0$ and $n = 2$, $\mathfrak{m}_{\lambda_0}$ is nondegenerate if and only if $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$.

Proof. The first statement is proved as in Corollary 3.10. When $\lambda_0 = 0$ and $n = 2$, due to (3.21), (3.33) reduces to

(3.34)
$$\mathfrak{M}_{\lambda_0} = -s_0 \begin{pmatrix} \langle \mathbf{u}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(1)} \rangle & \langle \mathbf{u}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(2)} \rangle \\ \langle \mathbf{u}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(1)} \rangle & \langle \mathbf{u}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(2)} \rangle \end{pmatrix} = -s_0 \begin{pmatrix} 0 & \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \\ \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle & 0 \end{pmatrix},$$

from which nondegeneracy of $\mathfrak{m}_{\lambda_0}$ occurs if and only if the condition stated holds. $\quad\square$

It follows from Corollaries 3.12 and 3.13 that a calculation of the Maslov index at $\lambda = 0$ in the λ-direction is not possible using the first-order crossing form (3.3) if $\dim \ker(N_{s_0}) = 1$ or if $\dim \ker(N_{s_0}) = 2$ and $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle = 0$. In light of this, we define the following.

DEFINITION 3.14. *The correction term $\mathfrak{c}$ is*

(3.35) $\mathfrak{c} := \mathrm{Mas}\big(\Lambda(s, \lambda), \mathcal{D}; s \in [1 - \varepsilon, 1]\big) + \mathrm{Mas}\big(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [0, \varepsilon]\big)$

for $0 < \varepsilon \ll 1$.

That is, $\mathfrak{c}$ denotes the contribution to the Maslov index from the top left corner of the Maslov box (consisting of the arrival along Γ_2 and the departure along Γ_3).

Remark 3.15. To see that this does not depend on the choice of $0 < \varepsilon \ll 1$, we observe that $(0, 1)$ is an isolated crossing for both Γ_2 and Γ_3. For Γ_2, this follows from the nondegeneracy of $\mathfrak{m}_{s_0}$ in Lemma 3.9 and Corollary 3.10. For Γ_3, we use the fact that the set $\{\lambda : \Lambda(\lambda, 1) \cap \mathcal{D} \neq \{0\}\} = \mathrm{Spec}(N) \cap \mathbb{R}$ is discrete (because N has compact resolvent), so there exists $\hat{\lambda} > 0$ such that $\Lambda(\lambda, 1) \cap \mathcal{D} = \{0\}$ for $0 < \lambda < \hat{\lambda}$.

We circumvent the issue of the nonregular crossing in subsection 4.4 via a homotopy argument. This will be possible after having analyzed the local behavior of the eigenvalue curves in section 4. In the meantime, we compute the second-order crossing form (3.7) from [23, Proposition 3.10].

LEMMA 3.16. *Assume the conditions of Lemma* 3.11. *If the first-order quadratic form in* (3.30) *is identically zero, then the second-order quadratic form* (3.7) *is given by*

(3.36) $$\mathfrak{m}_{\lambda_0}^{(2)}(q) = -2s_0^3 \langle \mathbf{v}_{s_0}, S\mathbf{u}_{s_0} \rangle, \qquad q = \mathrm{Tr}_{s_0} \mathbf{u}_{s_0},$$

where $\mathbf{u}_{s_0} \in \ker(N_{s_0} - s_0^2 \lambda_0)$ and $\mathbf{v}_{s_0} \in \mathrm{dom}(N_{s_0})$ solves $(N_{s_0} - s_0^2 \lambda_0)\mathbf{v}_{s_0} = \mathbf{u}_{s_0}$. The $n \times n$ matrix $\mathfrak{M}_{\lambda_0}^{(2)}$ of the symmetric bilinear form associated with $\mathfrak{m}_{\lambda_0}^{(2)}$ has entries

(3.37) $$\left[\mathfrak{M}_{\lambda_0}^{(2)}\right]_{ij} = -2s_0^3 \left\langle \mathbf{v}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle,$$

where $\mathbf{v}_{s_0}^{(i)} \in \mathrm{dom}(N_{s_0})$ solves $(N_{s_0} - s_0^2 \lambda_0)\mathbf{v}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$. In the case $\lambda_0 = 0$ and $n = 1$, we have

(3.38) $$\mathfrak{m}_{\lambda_0}^{(2)}(q) = \begin{cases} -2s_0^3 \left\langle \widehat{v}_{s_0}, u_{s_0} \right\rangle & 0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0}), \\ -2s_0^3 \left\langle \widehat{u}_{s_0}, v_{s_0} \right\rangle & 0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0}), \end{cases}$$

where $\widehat{v}_{s_0} \in \mathrm{dom}(L_-^{s_0})$ and $\widehat{u}_{s_0} \in \mathrm{dom}(L_+^{s_0})$ solve $-L_-^{s_0} \widehat{v}_{s_0} = u_{s_0}$ and $L_+^{s_0} \widehat{u}_{s_0} = v_{s_0}$, respectively. In the case $\lambda_0 = 0$ and $n = 2$, we have

(3.39) $$\mathfrak{M}_{\lambda_0}^{(2)} = -2s_0^3 \begin{pmatrix} \left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \right\rangle & 0 \\ 0 & \left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \right\rangle \end{pmatrix},$$

where $\widehat{v}_{s_0}^{(1)} \in \mathrm{dom}(L_-^{s_0})$ *and* $\widehat{u}_{s_0}^{(2)} \in \mathrm{dom}(L_+^{s_0})$ *solve* $-L_-^{s_0}\widehat{v}_{s_0}^{(1)} = u_{s_0}^{(1)}$ *and* $L_+^{s_0}\widehat{u}_{s_0}^{(2)} = v_{s_0}^{(2)}$, *respectively.*

Remark 3.17. The equation $(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$ is always solvable by virtue of the Fredholm alternative since $\mathfrak{m}_{s_0} = 0$ means $\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \rangle = 0$ for all i, j and hence implies $\mathbf{u}_{s_0}^{(i)}$ is orthogonal to $\ker(N_{s_0}^* - s_0^2\lambda_0)$. Such a solution is not unique; however, only the component of the solution in $\ker(N_{s_0} - s_0^2\lambda_0)^\perp$ (which is unique) contributes to (3.36). It therefore suffices to consider those $\mathbf{v}_{s_0}^{(i)}$ satisfying $\mathbf{v}_{s_0}^{(i)} \perp \mathbf{u}_{s_0}^{(j)}$ for all $j = 1, \ldots, n$. Notice that the $\mathbf{v}_{s_0}^{(i)}$ are generalized eigenfunctions: If $\mathfrak{m}_{\lambda_0} = 0$, the eigenvalue $s_0^2\lambda_0 \in \mathrm{Spec}(N_{s_0})$ has n Jordan chains of length (at least) two. We thus see that loss of regularity of the crossing coincides precisely with loss of semisimplicity of the eigenvalue, which agrees with the result of [15, Theorem 6.1].

Proof. Consider a C^2 family of vectors $\lambda \mapsto \mathbf{w}_\lambda$ satisfying (3.31). Then

$$\mathfrak{m}_{\lambda_0}^{(2)}(q) = \omega\left(\mathrm{Tr}_{s_0}\mathbf{u}_{s_0}, \mathrm{Tr}_{s_0}\ddot{\mathbf{w}}_\lambda\right)\big|_{\lambda=\lambda_0}.$$

Differentiating (3.31a) twice with respect to λ, applying $\langle \cdot, S\mathbf{w}_\lambda \rangle$, and rearranging yields

$$\langle (N_{s_0} - s_0^2\lambda)\ddot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle = 2s_0^2\langle \dot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle.$$

Now using (3.12) with $\mathbf{u} = \mathbf{w}_\lambda$ and $\mathbf{v} = \ddot{\mathbf{w}}_\lambda$, we have

$$s_0\,\omega(\mathrm{Tr}_{s_0}\mathbf{w}_\lambda, \mathrm{Tr}_{s_0}\ddot{\mathbf{w}}_\lambda) = \langle (N_{s_0} - s_0^2\lambda)\mathbf{w}_\lambda, S\ddot{\mathbf{w}}_\lambda \rangle - \langle S\mathbf{w}_\lambda, (N_{s_0} - s_0^2\lambda)\ddot{\mathbf{w}}_\lambda \rangle.$$

Combining (3.31a) with the previous two equations, we get

$$s_0\,\omega(\mathrm{Tr}_{s_0}\mathbf{w}_\lambda, \mathrm{Tr}_{s_0}\ddot{\mathbf{w}}_\lambda) = -2s_0^2\langle \dot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle.$$

Evaluating this last equation at $\lambda = \lambda_0$ and dividing through by s_0, we see that

$$\mathfrak{m}_{\lambda_0}^{(2)}(q) = \omega(\mathrm{Tr}_{s_0}\mathbf{u}_{s_0}, \mathrm{Tr}_{s_0}\ddot{\mathbf{w}}_\lambda)\big|_{\lambda=\lambda_0} = -2s_0\langle \dot{\mathbf{w}}_{\lambda_0}, S\mathbf{u}_{s_0} \rangle.$$

To compute $\dot{\mathbf{w}}_{\lambda_0}$, we see that differentiating (3.31a) with respect to λ, evaluating at $\lambda = \lambda_0$, and rearranging yields

(3.40) $$\left(N_{s_0} - s_0^2\lambda_0\right)\dot{\mathbf{w}}_{\lambda_0} = s_0^2\mathbf{u}_{s_0}.$$

Setting $s_0^2\mathbf{v}_{s_0} = \dot{\mathbf{w}}_{\lambda_0}$, (3.36) follows.

The same arguments as in the proof of Corollary 3.10 are used to prove (3.37). Equations (3.38) and (3.39) follow from the structure of the eigenvectors and generalized eigenvectors when $\lambda_0 = 0$. If $0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0})$ and $\widehat{u}_{s_0}$ is as stated in the lemma, we have

$$\begin{pmatrix} 0 & -L_-^{s_0} \\ L_+^{s_0} & 0 \end{pmatrix}\begin{pmatrix} \widehat{u}_{s_0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_{s_0} \end{pmatrix} = \mathbf{u}_{s_0},$$

so $\mathbf{v}_{s_0} = (\widehat{u}_{s_0}, 0)^\top$, and hence $\langle \mathbf{v}_{s_0}, S\mathbf{u}_{s_0} \rangle = \langle \widehat{u}_{s_0}, v_{s_0} \rangle$. If $0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0})$, we similarly find that $\mathbf{v}_{s_0} = (0, \widehat{v}_{s_0})^\top$, and hence $\langle \mathbf{v}_{s_0}, S\mathbf{u}_{s_0} \rangle = \langle \widehat{v}_{s_0}, u_{s_0} \rangle$. Finally, if $\dim\ker(N_{s_0}) = 2$, we have

(3.41) $$\mathbf{v}_{s_0}^{(1)} = \begin{pmatrix} 0 \\ \widehat{v}_{s_0}^{(1)} \end{pmatrix}, \quad \mathbf{v}_{s_0}^{(2)} = \begin{pmatrix} \widehat{u}_{s_0}^{(2)} \\ 0 \end{pmatrix},$$

with $\mathbf{u}_{s_0}^{(i)}$ given by (3.21). It follows that $\langle \mathbf{v}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(2)}\rangle = \langle \mathbf{v}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(1)}\rangle = 0$ and

$$(3.42) \qquad \langle \mathbf{v}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(1)}\rangle = \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)}\rangle, \qquad \langle \mathbf{v}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(2)}\rangle = \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)}\rangle,$$

which completes the proof. $\qquad\square$

Remark 3.18. The Maslov index is in general not monotone in λ in the sense that the form (3.30) is indefinite. Consequently, it does not necessarily give an exact count of the crossings along Γ_3 for $\lambda > 0$, which by Proposition 3.7 equals the number of real positive eigenvalues of N. Nonetheless, the Maslov index always provides a lower bound for this count, and this will be used in the proof of Theorem 2.2. In special cases it *is* possible to have monotonicity in λ; this will be used to obtain stability results in Theorem 2.11 (cf. Lemma 5.2).

3.4. Bounding the real eigenvalue count. Before proving Theorem 2.2, we list some preliminary results. The first is a version of the Morse index theorem (see [64, section 15], [73]) for scalar-valued Schrödinger operators on bounded domains with Dirichlet boundary conditions. Recall that the Morse indices P and Q are the numbers of negative eigenvalues of the operators L_+ and L_-, respectively.

LEMMA 3.19. *The Morse index of L_+ equals the number of conjugate points for L_+ in $(0,1)$,*

$$(3.43) \qquad P = \#\{s_0 \in (0,1) : 0 \in \mathrm{Spec}(L_+^{s_0})\},$$

and likewise for L_- and Q.

The following lemma will not be needed until the proof of Lemma 5.1, but we list it here since its proof uses the same ideas that are used to prove the previous lemma.

LEMMA 3.20. *If $Q = 0$ (resp., $P = 0$), then L_-^s (resp., L_+^s) is a strictly positive operator for all $s \in (0,1)$ and is nonnegative for $s = 1$.*

Proof. This follows from monotonicity of the eigenvalues of the Schrödinger operators $L_\pm^s$ in the spatial parameter s; see [73]. Indeed, the eigenvalues $\lambda_j^\pm(s) \in \mathrm{Spec}(L_\pm^s)$ are strictly decreasing functions of s, so $\lambda_j^\pm(1) \geq 0$ implies $\lambda_j^\pm(s) > 0$ for $s \in (0,1)$. $\quad\square$

The following self-adjoint formulation of the eigenvalue problem will be needed in Lemma 3.23. Some of the ideas used here, especially the use of the square root of a strictly positive operator to convert the eigenvalue problem to a self-adjoint one, can be found in [66, section 4].

LEMMA 3.21. *Fix $s \in (0,1]$, and suppose $\lambda \in \mathbb{C}\backslash\{0\}$. If L_-^s is a nonnegative operator, the eigenvalue problem*

$$(3.44) \qquad \begin{cases} \textit{There exists } v_s \in \mathrm{dom}(L_-^s), u_s \in \mathrm{dom}(L_+^s) \textit{ such that}: \\ -L_-^s v_s = s^2\lambda u_s, \quad L_+^s u_s = s^2\lambda v_s \end{cases}$$

is equivalent to

$$(3.45) \qquad \begin{cases} \textit{There exists } w_s \in \mathrm{dom}\left(L_-^s|_{X_c}\right)^{1/2} \textit{ with } \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_+^s) \\ \textit{and } L_+^s\Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_-^s), \textit{ such that}: \\ \left(L_-^s|_{X_c}\right)^{1/2}\Pi L_+^s\Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = -s^4\lambda^2 w_s, \end{cases}$$

where the domains $\mathrm{dom}(L_\pm^s)$ are given by (2.5), $X_c := \ker(L_-^s)^\perp \subseteq L^2(0,\ell)$, and Π is the orthogonal projection $\Pi : L^2(0,\ell) \to X_c$. If L_+^s is nonnegative, then (3.44) is equivalent to

$$(3.46) \quad \begin{cases} \textit{There exists } w_s \in \mathrm{dom}\left(L_+^s|_{X_c}\right)^{1/2} \textit{ with } \Pi\left(L_+^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_-^s) \\ \textit{and } L_-^s \Pi\left(L_+^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_+^s), \textit{ such that}: \\ \left(L_+^s|_{X_c}\right)^{1/2} \Pi L_-^s \Pi\left(L_+^s|_{X_c}\right)^{1/2} w_s = -s^4 \lambda^2 w_s, \end{cases}$$

where now $X_c := \mathrm{ker}(L_+^s)^\perp \subseteq L^2(0,\ell)$.

Proof. We begin with the case $L_-^s \geq 0$. We prove the equivalence of (3.44) and (3.45) via their equivalence with

$$(3.47) \quad \begin{cases} \text{There exists } u_s \in \mathrm{dom}(L_+^s) \cap X_c \text{ with } L_+^s u_s \in \mathrm{dom}(L_-^s), \text{ such that}: \\ L_-^s L_+^s u_s = -s^4 \lambda^2 u_s. \end{cases}$$

Defining the restricted operator $L_-^s|_{X_c}$ acting in X_c by

$$L_-^s|_{X_c} v := L_-^s v, \qquad v \in \mathrm{dom}(L_-^s|_{X_c}) := \mathrm{dom}(L_-^s) \cap X_c,$$

note that $L_-^s|_{X_c} > 0$ and $\left(L_-^s|_{X_c}\right)^{1/2}$ is a well-defined and invertible operator acting in X_c.

(3.44) $\implies$ (3.47): Clearly, $L_+^s u_s = s^2 \lambda v_s \in \mathrm{dom}(L_-^s)$, and $u_s = -\frac{1}{s^2\lambda} L_-^s v_s \in \mathrm{ran}\, L_-^s = X_c$ because L_-^s is self-adjoint and Fredholm. Applying L_-^s to the second equation in (3.44) yields the equation in (3.47).

(3.47) $\implies$ (3.45): Set $w_s := \left(L_-^s|_{X_c}\right)^{-1/2} u_s$. Then $w_s \in \mathrm{dom}\left(L_-^s|_{X_c}\right)^{1/2}$, and since $u_s \in X_c$, we have $\Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = \Pi u_s = u_s \in \mathrm{dom}(L_+^s)$, and $L_+^s \Pi u_s = L_+^s u_s \in \mathrm{dom}(L_-^s)$. Now $L_+^s u_s = \Pi L_+^s u_s + (I-\Pi) L_+^s u_s$, where the projection $(I - \Pi) : L^2(0,\ell) \to \mathrm{ker}(L_-^s) \subset \mathrm{dom}(L_-^s)$. Then $\Pi L_+^s u_s \in \mathrm{dom}(L_-^s) \cap X_c = \mathrm{dom}(L_-^s|_{X_c})$. Thus, $L_-^s L_+^s u_s = L_-^s \Pi L_+^s \Pi u_s = L_-^s|_{X_c} \Pi L_+^s \Pi u_s = \left(L_-^s|_{X_c}\right)^{1/2} \left(L_-^s|_{X_c}\right)^{1/2} \Pi L_+^s \Pi u_s$. Substituting this into the equation in (3.47) and multiplying by $\left(L_-^s|_{X_c}\right)^{-1/2}$ gives the equation in (3.45).

(3.45) $\implies$ (3.44): Set $u_s := \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_+^s)$ and $v_s := \frac{1}{s^2\lambda} L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_-^s)$. Then $L_+^s u_s = L_+^s \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = s^2 \lambda v_s$, and since Π projects onto $\mathrm{ran}(L_-^s)$, $-L_-^s v_s = -\Pi L_-^s v_s = \frac{-1}{s^2\lambda} \Pi L_-^s L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s = \frac{-1}{s^2\lambda} \Pi L_-^s (\Pi + (I-\Pi) L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s = \frac{-1}{s^2\lambda} \Pi L_-^s \Pi L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s = s^2 \lambda \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s = s^2 \lambda u_s$.

The case $L_+^s \geq 0$ uses similar arguments, except now (3.44) and (3.46) are equivalent via

$$\begin{cases} \text{There exists } v_s \in \mathrm{dom}(L_-^s) \cap X_c \text{ with } L_-^s v_s \in \mathrm{dom}(L_+^s), \text{ such that}: \\ L_+^s L_-^s v_s = -s^4 \lambda^2 v_s. \end{cases}$$

We omit the details. $\qquad \square$

We are now ready to compute the Maslov index of Γ_2^ε, the restriction of Γ_2 to $[\tau, 1-\varepsilon]$.

LEMMA 3.22. *The Maslov index of the Lagrangian path $s \mapsto \Lambda(0,s) \subset \mathbb{R}^8$, $s \in [\tau, 1-\varepsilon]$, is*

$$(3.48) \qquad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) = Q - P.$$

Proof. Consider the crossing form

$$\mathfrak{m}_{s_0}(q) = \frac{\ell}{s_0^2}\left[-\left(u'_{s_0}(\ell)\right)^2 + \left(v'_{s_0}(\ell)\right)^2\right]$$

from (3.23) and recall (3.20). If $(0, s_0)$ is a simple crossing, we obtain $\mathfrak{m}_{s_0} < 0$ if $0 \in \mathrm{Spec}(L_+^{s_0})$ and $\mathfrak{m}_{s_0} > 0$ if $0 \in \mathrm{Spec}(L_-^{s_0})$. On the other hand, if $0 \in \mathrm{Spec}(L_+^{s_0}) \cap \mathrm{Spec}(L_-^{s_0})$, the 2×2 matrix $\mathfrak{M}_{s_0}$ in (3.29) has eigenvalues of opposite sign, so we conclude that

$$(3.49) \qquad \mathrm{sign}\,(\mathfrak{m}_{s_0}) = \begin{cases} -1 & 0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0}), \\ +1 & 0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0}), \\ 0 & 0 \in \mathrm{Spec}(L_+^{s_0}) \cap \mathrm{Spec}(L_-^{s_0}). \end{cases}$$

From the definition (3.4), we then have

$$\begin{aligned} \mathrm{Mas}(\Lambda(0, s), \mathcal{D}; s \in [\tau, 1-\varepsilon]) &= -\#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0})\} \\ &\quad + \#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0})\} \\ &= -\#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_+^{s_0})\} \\ &\quad + \#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_-^{s_0})\}, \end{aligned}$$

and the result follows using Lemma 3.19. □

Next, we prove that there are no crossings along Γ_1 and Γ_4; we refer the reader to Figure 2.

LEMMA 3.23. $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_1) = \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0$ *provided $\tau > 0$ is sufficiently small and $\lambda_\infty > 0$ is sufficiently large.*

Proof. For the case of no crossings along Γ_1, we prove that N_s has no real eigenvalues for $s = \tau$ small enough. Seeking a contradiction, assume there exists $\tau^2\lambda \in \mathrm{Spec}(N_\tau) \cap \mathbb{R}$ with eigenfunction $\mathbf{u}_\tau = (u_\tau, v_\tau)^\top$.

First, note that the operators $L_\pm^\tau$ with domains given by (2.5) are strictly positive: By the Poincaré and Cauchy–Schwarz inequalities,

$$\langle L_+^\tau v, v\rangle = \|v'\|^2 - \langle \tau^2 g(\tau x)v, v\rangle \geq C\|v\|^2 - \tau^2\|g\|_\infty\|v\|^2$$

for some $C > 0$ and all $v \in \mathrm{dom}(L_+^\tau)$, so we choose τ small enough that $C > \tau^2\|g\|_\infty$. Due to the decoupling of the eigenvalue equations for N_τ when $\lambda = 0$, it follows that $0 \notin \mathrm{Spec}(N_\tau)$.

Next, for $\lambda \in \mathbb{R}\setminus\{0\}$, we note that by Lemma 3.21, the eigenvalue equations for N_τ are equivalent to

$$(3.50) \qquad \left(L_-^\tau\right)^{1/2} L_+^\tau \left(L_-^\tau\right)^{1/2} w_\tau = -\tau^4\lambda^2 w_\tau$$

since the positivity of L_-^τ implies that $X_c = \ker(L_-^\tau)^\perp$ is all of $L^2(0, \ell)$, and hence the resulting projection Π is the identity. Applying $\langle\cdot, w_\tau\rangle$ to (3.50), we immediately see that the right-hand side is negative, while for the left-hand side, we obtain

$$\begin{aligned} \langle\left(L_-^\tau\right)^{1/2} L_+^\tau \left(L_-^\tau\right)^{1/2} w_\tau, w_\tau\rangle &= \langle L_+^\tau \left(L_-^\tau\right)^{1/2} w_\tau, \left(L_-^\tau\right)^{1/2} w_\tau\rangle \\ &\geq C_+\langle\left(L_-^\tau\right)^{1/2} w_\tau, \left(L_-^\tau\right)^{1/2} w_\tau\rangle \\ &= C_+\langle L_-^\tau w_\tau, w_\tau\rangle \\ &\geq C_+ C_-\|w_\tau\|^2 > 0 \end{aligned}$$

for some positive constants $C_\pm$ (using the positivity of $L_\pm^\tau$ and self-adjointness of $\left(L_-^\tau\right)^{1/2}$), a contradiction. We conclude that no such real $\tau^2\lambda \in \mathrm{Spec}(N_\tau)$ exists, and there are no crossings along Γ_1.

Moving to Γ_4, we show that the spectrum of N_s lies in a vertical strip around the imaginary axis in the complex plane for all $s \in (0, 1]$). For this, it suffices to show that $\mathrm{Spec}(iN_s)$ lies in a horizontal strip around the real axis since $\mathrm{Spec}(iN_s) = -i\,\mathrm{Spec}(iN_s)$ by the spectral mapping theorem. Fixing $s \in (0, 1]$, we have

$$(3.51) \qquad\qquad iN_s = iD + iB_s(x),$$

where iD is self-adjoint and $iB_s(x)$ is bounded. It then follows from [54, Remark 3.2, page 208] and [54, equation (3.16), page 272] that

$$(3.52) \qquad\qquad \zeta \in \mathrm{Spec}(iD + iB_s(x)) \implies |\mathrm{Im}\,\zeta| \leq \|iB_s(x)\|,$$

as required. Choosing $\lambda_\infty > \sup_{s\in(0,1]}\|B_s(x)\|$ ensures there are no crossings along Γ_4. □

We are our ready to prove our first main result.

Proof of Theorem 2.2. As already observed in (3.18), the homotopy invariance and additivity of the Maslov index yield

$$(3.53) \qquad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_1) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0;$$

hence,

$$(3.54) \qquad\qquad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3) = 0$$

by Lemma 3.23. Again using additivity and recalling the definition of $\mathfrak{c}$ in Definition 3.14, we rewrite this as

$$(3.55) \qquad\qquad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) + \mathfrak{c} + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = 0,$$

where Γ_2^ε was defined in Lemma 3.22 and Γ_3^ε is the restriction of Γ_3 to $[\varepsilon, \lambda_\infty]$. Using Lemma 3.22, we thus obtain

$$(3.56) \qquad\qquad \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = P - Q - \mathfrak{c}.$$

As discussed in Remark 3.18, the lack of monotonicity in λ means that $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$ does not necessarily count the number of real, positive eigenvalues of N. Nonetheless, we still have that

$$(3.57) \qquad\qquad n_+(N) \geq |\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)|,$$

and (2.7) follows. □

4. The eigenvalue curves. In this section, we analyze the real eigenvalue curves of N_s in the λs-plane. We consider the general case of a crossing (λ_0, s_0) corresponding to an eigenvalue $s_0^2\lambda_0 \in \mathrm{Spec}(N_{s_0})$ with $\dim\ker(N_{s_0} - s_0^2\lambda_0) = n$, paying special attention to the cases $\lambda_0 = 0$ and $n = 1, 2$. We use the results obtained to compute the correction term $\mathfrak{c}$ from Theorem 2.2, and relate a component of it to the signature of the second-order crossing form (3.36) in Proposition 4.15.

4.1. Numerical description. We begin with a brief description of a tool that is useful for numerically computing the eigenvalue curves. The idea is to globally characterize the set of points (λ, s) such that $s^2 \lambda \in \operatorname{Spec}(N_s) \cap \mathbb{R}$ as the zero set of a function called the *characteristic determinant.*

Converting the restricted problem (2.6) with $y \in [0, s\ell]$ to a first-order system yields

$$(4.1) \qquad \frac{d}{dy} \begin{pmatrix} u \\ v \\ r \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -g(y) & -\lambda & 0 & 0 \\ -\lambda & h(y) & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ r \\ z \end{pmatrix}.$$

Notice that we use the substitution $\partial_y v = -z$ in order to preserve the Hamiltonian structure. Rescaling as in subsection 3.2, we define $u_s(x) := u(sx)$ for $x \in [0, \ell]$ and similarly for v_s, r_s, and z_s. Then the equivalent system on $[0, \ell]$ is

$$(4.2) \qquad \frac{d}{dx} \begin{pmatrix} u_s \\ v_s \\ r_s \\ z_s \end{pmatrix} = \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & -s \\ -sg(sx) & -s\lambda & 0 & 0 \\ -s\lambda & sh(sx) & 0 & 0 \end{pmatrix} \begin{pmatrix} u_s \\ v_s \\ r_s \\ z_s \end{pmatrix}.$$

Consider a fundamental matrix solution $\Phi(x; \lambda, s) \in \mathbb{R}^{4\times 4}$ to (4.2) with $\Phi(0; \lambda, s) = I_4$. For convenience, we write Φ as the block matrix

$$\Phi(x; \lambda, s) = \begin{pmatrix} U(x; \lambda, s) & X(x; \lambda, s) \\ V(x; \lambda, s) & Y(x; \lambda, s) \end{pmatrix}, \qquad U, V, X, Y \in \mathbb{R}^{2\times 2},$$

where

$$(4.3) \qquad U(0; \lambda, s) = Y(0; \lambda, s) = I_2, \qquad V(0; \lambda, s) = X(0; \lambda, s) = 0_2.$$

Because Φ is a matrix solution for (4.2), we have

$$(4.4) \quad \frac{d}{dx} \begin{pmatrix} U & X \\ V & Y \end{pmatrix} = \begin{pmatrix} 0 & s\sigma_3 \\ s\left(SB(sx) - \lambda S\right) & 0 \end{pmatrix} \begin{pmatrix} U & X \\ V & Y \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

PROPOSITION 4.1. *For all $(\lambda, s) \in \mathbb{R} \times (0, 1]$, the following are equivalent:*
1. $\lambda \in \operatorname{Spec}(N|_{[0, s\ell]}) \cap \mathbb{R}$;
2. $s^2 \lambda \in \operatorname{Spec}(N_s) \cap \mathbb{R}$;
3. $\Lambda(\lambda, s) \cap \mathcal{D} \neq \{0\}$;
4. $\det X(\ell; \lambda, s) = 0$.

We thus call $\det X(\ell; \lambda, s)$ the *characteristic determinant*: The real eigenvalue curves in the λs-plane are given by the zero set $\{(\lambda, s) : \det X(\ell; \lambda, s) = 0\}$. Figure 3 illustrates some examples of these curves under Hypothesis 2.5.

Proof. The discussion following (3.11) gives the equivalence of (1) and (2), while the equivalence of (2) and (3) was given in Proposition 3.7. We show the equivalence of (3) and (4). Fix $s \in (0, 1]$ and $\lambda \in \mathbb{R}$, and consider the 8×4 matrix

$$\mathcal{Z}(\lambda, s) := \begin{pmatrix} U(0; \lambda, s) & X(0; \lambda, s) \\ U(\ell; \lambda, s) & X(\ell; \lambda, s) \\ -V(0; \lambda, s) & -Y(0; \lambda, s) \\ V(\ell; \lambda, s) & Y(\ell; \lambda, s) \end{pmatrix} = \begin{pmatrix} I_2 & 0_2 \\ U(\ell; \lambda, s) & X(\ell; \lambda, s) \\ 0_2 & -I_2 \\ V(\ell; \lambda, s) & Y(\ell; \lambda, s) \end{pmatrix}.$$
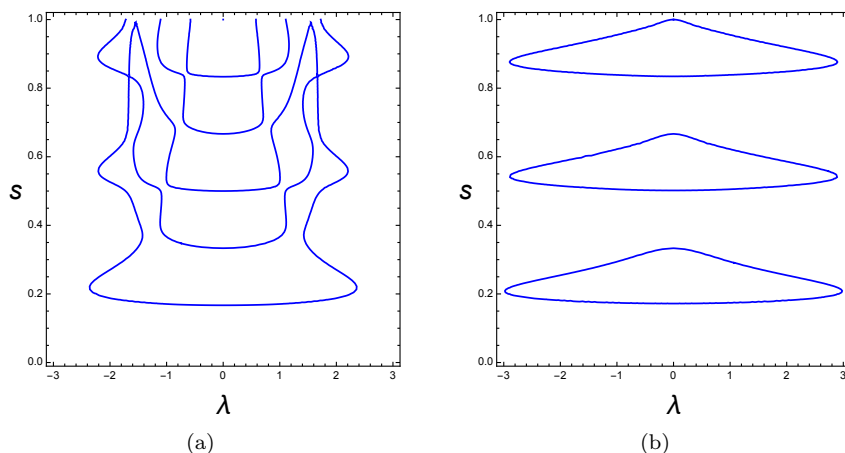
(a)   (b)

FIG. 3. *Real eigenvalue curves* $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$ *under Hypothesis* 2.5(i) *associated with a T-periodic stationary state* ϕ_0 *with nonlinearity* $f(\phi^2) = \phi^2$ *and* $\beta = -2$. *In* (a), ϕ_0 *is a positive Jacobi dnoidal function (i.e., an orbit located inside the homoclinic orbit in the right half plane in Figure* 1(a)) *satisfying* $\phi_0'(0) = \phi_0'(\ell) = 0$ *with* $\ell = 3T = 9.9398$. *In* (b), ϕ_0 *is a Jacobi cnoidal function (i.e., an orbit located outside the homoclinic orbit in Figure* 1(a)) *satisfying* $\phi_0(0) = \phi_0(\ell) = 0$ *with* $\ell = 3T/2 = 10.0391$.

Notice that the columns of $\mathcal{Z}(\lambda, s)$ are precisely the rescaled trace (cf. (3.10)) of four linearly independent functions in $\mathcal{K}_{\lambda,s}$ (recall that the entries of $Y(\cdot; \lambda, s)$ and $V(\cdot; \lambda, s)$ satisfy $r_s = s^{-1}\partial_x u_s$ and $z_s = -s^{-1}\partial_x v_s$) and thus are a basis for our Lagrangian subspace $\Lambda(\lambda, s)$.

A nontrivial intersection of the four-dimensional linear subspaces $\Lambda(\lambda, s)$ and $\mathcal{D}$ of $\mathbb{R}^8$ occurs if and only if the 8×8 matrix formed by their bases has zero determinant. Therefore,

$$\Lambda(\lambda, s) \cap \mathcal{D} \neq \{0\} \iff \det \begin{pmatrix} I & 0 & 0 & 0 \\ U(\ell; \lambda, s) & X(\ell; \lambda, s) & 0 & 0 \\ -0 & -I & I & 0 \\ V(\ell; \lambda, s) & Y(\ell; \lambda, s) & 0 & I \end{pmatrix} = 0 \iff \det X(\ell; \lambda, s) = 0,$$

as required. $\qquad\square$

4.2. Analytic description. We will generalize Theorem 2.9 to Theorem 4.5, which is a consequence of the following general result. We remind the reader that $n \leq 2$ in the current paper; see Remark 3.8. Below, dot denotes $d/d\lambda$.

PROPOSITION 4.2. *Assume* $\dim \ker(N_{s_0} - s_0^2\lambda_0) = n$ *with basis* $\{\mathbf{u}_{s_0}^{(1)}, \ldots, \mathbf{u}_{s_0}^{(n)}\}$. *There exists an* $n \times n$ *matrix* $M(\lambda, s)$, *defined near* (λ_0, s_0), *such that* $s^2\lambda \in \mathrm{Spec}(N_s)$ *if and only if* $\det M(\lambda, s) = 0$. *This matrix satisfies* $M(\lambda_0, s_0) = 0$ *and*

(4.5)
$$\frac{\partial M_{ij}}{\partial \lambda}(\lambda_0, s_0) = -s_0^2 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \quad \frac{\partial M_{ij}}{\partial s}(\lambda_0, s_0) = \left\langle \left(\partial_s B_{s_0} - 2s_0\lambda_0\right)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle.$$

Moreover, if $\left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0$ *for all* $i, j = 1, \ldots, n$, *then*

(4.6)
$$\frac{\partial^2 M_{ij}}{\partial \lambda^2}(\lambda_0, s_0) = -2s_0^4 \left\langle \mathbf{v}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle,$$

where $\mathbf{v}_{s_0}^{(i)} \in \mathrm{dom}(N_{s_0})$ *solves the inhomogeneous equation* $(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$.

Remark 4.3. Just as in Remark 3.17, for (4.6), it suffices to consider those solutions to the inhomogeneous equation that satisfy $\mathbf{v}_{s_0}^{(i)} \perp \mathbf{u}_{s_0}^{(j)}$ for $i, j = 1, \ldots, n$.

The definition of M, which requires some preparation, is given in (4.14).

Proof. We construct $M(\lambda, s)$ using Lyapunov–Schmidt reduction. The first step is to split the eigenvalue equation $(N_s - s^2\lambda)\mathbf{u} = 0$ into two parts, one of which can always be solved uniquely. Let P denote the L^2-orthogonal projection onto $\ker(N_{s_0}^* - s_0^2\lambda_0)$, so that $I - P$ is the projection onto $\ker(N_{s_0}^* - s_0^2\lambda_0)^\perp = \operatorname{ran}(N_{s_0} - s_0^2\lambda_0)$. It follows that $s^2\lambda$ is an eigenvalue of N_s if and only if there exists a nonzero $\mathbf{u} \in \operatorname{dom}(N_s)$ such that both

$$P(N_s - s^2\lambda)\mathbf{u} = 0 \tag{4.7}$$

and

$$(I - P)(N_s - s^2\lambda)\mathbf{u} = 0 \tag{4.8}$$

hold.

We first consider (4.8). Defining $X_0 = \ker(N_{s_0} - s_0^2\lambda_0)^\perp \cap H^2(0, \ell) \cap H_0^1(0, \ell)$, we have that any $\mathbf{u} \in H^2(0, \ell) \cap H_0^1(0, \ell)$ can be written uniquely as

$$\mathbf{u} = \sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)} + \tilde{\mathbf{u}},$$

where $t_i \in \mathbb{R}$ and $\tilde{\mathbf{u}} \in X_0$. This means (4.8) holds if and only if there exists a vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and a function $\tilde{\mathbf{u}} \in X_0$ such that

$$(I - P)(N_s - s^2\lambda)\left(\sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)} + \tilde{\mathbf{u}}\right) = 0. \tag{4.9}$$

We claim that for each $(\mathbf{t}, \lambda, s)$, there exists a unique $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\mathbf{t}, \lambda, s) \in X_0$ satisfying (4.9). Writing this equation out explicitly, it is

$$(I - P)(N_s - s^2\lambda)\tilde{\mathbf{u}}(\mathbf{t}, \lambda, s) = -(I - P)(N_s - s^2\lambda)\sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)}.$$

We define

$$T(\lambda, s) : X_0 \to \operatorname{ran}(N_{s_0} - s_0^2\lambda_0), \qquad T(\lambda, s) = (I - P)\left(N_s - s^2\lambda\right)\Big|_{X_0}$$

and observe that $T(\lambda_0, s_0)$ is invertible; hence, $T(\lambda, s)$ is also invertible for nearby (λ, s). Defining

$$A(\lambda, s) : X_0^\perp \to X_0, \qquad A(\lambda, s) = -T^{-1}(\lambda, s)(I - P)\left(N_s - s^2\lambda\right)\Big|_{X_0^\perp},$$

where $X_0^\perp = \ker(N_{s_0} - s_0^2\lambda_0)$, the unique solution to (4.9) is thus

$$\tilde{\mathbf{u}}(\mathbf{t}, \lambda, s) = A(\lambda, s)\sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)}. \tag{4.10}$$

So far, we have shown that the equation $(I - P)(N_s - s^2\lambda)\mathbf{u} = 0$ is satisfied if and only if $\mathbf{u}$ has the form

$$(4.11) \qquad \mathbf{u} = \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} + A(\lambda, s) \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} = \big(I + A(\lambda, s)\big) \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)}$$

for some $\mathbf{t} \in \mathbb{R}^n$. We conclude that there exists $\mathbf{u}$ for which $(N_s - s^2\lambda)\mathbf{u} = 0$ holds if and only if

$$(4.12) \qquad P(N_s - s^2\lambda)\big(I + A(\lambda, s)\big) \left(\sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} \right) = 0$$

for some $\mathbf{t} \in \mathbb{R}^n$. Moreover, $\mathbf{u}$ is nonzero if and only if $\mathbf{t}$ is nonzero. Finally, we observe that $\ker(N_{s_0}^* - s_0^2\lambda_0)$ is spanned by $\{S\mathbf{u}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(2)}, \ldots, S\mathbf{u}_{s_0}^{(n)}\}$, and so (4.12) is equivalent to

$$(4.13) \qquad \left\langle (N_s - s^2\lambda)\big(I + A(\lambda, s)\big) \left(\sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} \right), S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0, \qquad j = 1, \ldots, n.$$

Defining the $n \times n$ matrix $M(\lambda, s)$ by

$$(4.14) \qquad M_{ij}(\lambda, s) = \left\langle (N_s - s^2\lambda)\big(I + A(\lambda, s)\big) \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \quad i, j = 1, \ldots, n,$$

the system of n equations (4.13) may be written as $M(\lambda, s)\mathbf{t} = 0$, which is satisfied for a nonzero vector $\mathbf{t}$ if and only if $\det M(\lambda, s) = 0$. This completes the first part of the proof.

It follows that $M(\lambda_0, s_0) = 0$. We then compute

$$(4.15)$$
$$\frac{\partial M_{ij}}{\partial \lambda}(\lambda_0, s_0) = \left\langle -s_0^2\big(I + A(\lambda_0, s_0)\big)\mathbf{u}_{s_0}^{(i)} + (N_{s_0} - s_0^2\lambda_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle$$
$$(4.16) \qquad\qquad = -s_0^2 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle,$$

where in the second line, we have used the fact that $A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)} = 0$ and

$$\left\langle (N_{s_0} - s_0^2\lambda_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle = \left\langle \partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, (N_{s_0}^* - s_0^2\lambda_0)S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0$$

because $S\mathbf{u}_{s_0}^{(j)} \in \ker(N_{s_0}^* - s_0^2\lambda_0)$. The s derivative is computed similarly.

Finally, if $\partial_\lambda M(\lambda_0, s_0) = 0$, we have

$$(4.17) \qquad \frac{\partial^2 M_{ij}}{\partial \lambda^2}(\lambda_0, s_0) = -2s_0^2 \left\langle \partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle,$$

where $\left\langle (N_{s_0} - s_0^2\lambda_0)\partial_{\lambda\lambda} A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0$ again using $S\mathbf{u}_{s_0}^{(j)} \in \ker(N_{s_0}^* - s_0^2\lambda_0)$. To compute $\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}$, we use the definition of $A(\lambda, s)$ to write

$$T(\lambda, s)A(\lambda, s)\mathbf{u}_{s_0}^{(i)} = -(I - P)\big(N_s - s^2\lambda\big)\mathbf{u}_{s_0}^{(i)}.$$

Differentiating in λ and again using the fact that $A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)} = 0$, we get

$$T(\lambda_0, s_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)} = s_0^2(I - P)\mathbf{u}_{s_0}^{(i)}.$$

The fact that $\left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0$ for all i,j implies $(I-P)\mathbf{u}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$. Setting $s_0^2 \mathbf{v}_{s_0}^{(i)} = \partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}$, we see from the definition of T that

$$T(\lambda_0, s_0)(s_0^2 \mathbf{v}_{s_0}^{(i)}) = s_0^2(I-P)(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = s_0^2(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)},$$

and the result follows. $\qquad\square$

Comparison with the symmetric matrices (3.27), (3.33), and (3.37) associated with the first- and second-order crossing forms reveals that the partial derivatives of the matrix M satisfy

$$(4.18) \qquad \frac{\partial M}{\partial s}(\lambda_0, s_0) = s_0\,\mathfrak{M}_{s_0}, \quad \frac{\partial M}{\partial \lambda}(\lambda_0, s_0) = s_0\,\mathfrak{M}_{\lambda_0}, \quad \frac{\partial^2 M}{\partial \lambda^2}(\lambda_0, s_0) = s_0\,\mathfrak{M}_{\lambda_0}^{(2)},$$

where the last formula holds when $\partial_\lambda M(\lambda_0, s_0) = 0$. In particular, in the case $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$ (so that M is a scalar), we have

(4.19)
$$\frac{\partial M}{\partial s}(\lambda_0, s_0) = s_0\,\mathfrak{m}_{s_0}(q), \quad \frac{\partial M}{\partial \lambda}(\lambda_0, s_0) = s_0\,\mathfrak{m}_{\lambda_0}(q), \quad \frac{\partial^2 M}{\partial \lambda^2}(\lambda_0, s_0) = s_0\mathfrak{m}_{\lambda_0}^{(2)}(q),$$

where again the last formula holds when $\partial_\lambda M(\lambda_0, s_0) = 0$. Combining (4.19) with the implicit function theorem immediately yields the following Hadamard-type formulas for the derivatives of the real eigenvalue curves in terms of the crossing forms.

COROLLARY 4.4. *Under the assumption that* $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$, *the following hold:*

1. *If* $\mathfrak{m}_{\lambda_0} \neq 0$, *then there exists a* C^2 *curve* $\lambda(s)$ *near* s_0 *such that*

$$(4.20) \qquad \lambda'(s_0) = -\frac{\mathfrak{m}_{s_0}(q)}{\mathfrak{m}_{\lambda_0}(q)}.$$

2. *If* $\mathfrak{m}_{s_0} \neq 0$, *then there exists a* C^2 *curve* $s(\lambda)$ *near* λ_0 *such that*

$$(4.21) \qquad \dot{s}(\lambda_0) = -\frac{\mathfrak{m}_{\lambda_0}(q)}{\mathfrak{m}_{s_0}(q)}.$$

Moreover, $\dot{s}(\lambda_0) = 0$ *if and only if* $\mathfrak{m}_{\lambda_0}(q) = 0$, *and in this case,*

$$(4.22) \qquad \ddot{s}(\lambda_0) = -\frac{\mathfrak{m}_{\lambda_0}^{(2)}(q)}{\mathfrak{m}_{s_0}(q)}.$$

Using this, we can construct a curve $s(\lambda)$ through any simple conjugate point and determine its concavity by an explicit formula.

THEOREM 4.5. *If* $\dim\ker N_{s_0} = 1$, *then for* $|\lambda| \ll 1$, *there exists a* C^2 *curve* $s(\lambda)$ *such that* $s(\lambda)^2\lambda \in \mathrm{Spec}(N_{s(\lambda)})$ *and a continuous curve* $\mathbf{u}_{s(\lambda)}$ *of eigenfunctions such that* $\mathbf{u}_{s(\lambda)} \to \mathbf{u}_{s_0}$ *as* $\lambda \to 0$. *Moreover,* $s(0) = s_0$, $\dot{s}(0) = 0$, *and the concavity of* $s(\lambda)$ *can be determined as follows:*

1. *If* $0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0})$ *with eigenfunction* $v_{s_0} \in \ker L_-^{s_0}$, *then*

$$(4.23) \qquad \ddot{s}(0) = \frac{2s_0^5}{\ell}\frac{\langle \widehat{u}_{s_0}, v_{s_0}\rangle}{\left(v_{s_0}'(\ell)\right)^2},$$

where $\widehat{u}_{s_0} \in H^2(0,\ell) \cap H_0^1(0,\ell)$ *is the unique solution to* $L_+^{s_0}\widehat{u}_{s_0} = v_{s_0}$.

2. *If $0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0})$ with eigenfunction $u_{s_0} \in \ker L_+^{s_0}$, then*

$$(4.24) \qquad \ddot{s}(0) = -\frac{2s_0^5}{\ell} \frac{\langle \widehat{v}_{s_0}, u_{s_0} \rangle}{\left(u'_{s_0}(\ell)\right)^2},$$

where $\widehat{v}_{s_0} \in H^2(0,\ell) \cap H_0^1(0,\ell)$ is the unique solution to $-L_-^{s_0} \widehat{v}_{s_0} = u_{s_0}$.

Proof. Lemma 3.9 implies $\mathfrak{m}_{s_0} \neq 0$, so the existence of $s(\lambda)$ follows from Corollary 4.4. Corollary 3.12 then gives $\dot{s}(0) = 0$. From (4.11), we see that $\mathbf{u}_{s(\lambda)} = \left(I + A(\lambda, s(\lambda))\right)\mathbf{u}_{s_0}$ is an eigenfunction of $N_{s(\lambda)}$ for the eigenvalue $s^2(\lambda)\lambda$. Since $A(\lambda, s(\lambda))$ is continuous in λ and $A(0, s_0)\mathbf{u}_{s_0} = 0$, the convergence of $\mathbf{u}_{s(\lambda)}$ to $\mathbf{u}_{s_0}$ follows.

It thus remains to prove (4.23) and (4.24). If $0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0})$, then u_{s_0} is trivial, so (3.23) and (3.38) give

$$(4.25) \qquad \mathfrak{m}_{s_0}(q) = \frac{\ell}{s_0^2} \left(v'_{s_0}(\ell)\right)^2, \qquad \mathfrak{m}_{\lambda_0}^{(2)}(q) = -2s_0^3 \langle \widehat{u}_{s_0}, v_{s_0} \rangle.$$

Substituting these into (4.22) immediately gives (4.23). The case $0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0})$ is almost identical. Here we have

$$\mathfrak{m}_{s_0}(q) = -\frac{\ell}{s_0^2} \left(u'_{s_0}(\ell)\right)^2, \qquad \mathfrak{m}_{\lambda_0}^{(2)}(q) = -2s_0^3 \langle \widehat{v}_{s_0}, u_{s_0} \rangle,$$

and (4.24) follows. $\qquad \square$

4.3. When $\lambda_0 = 0$ has geometric multiplicity two. In this section. we focus on the case of a geometrically double eigenvalue at zero. Since $0 \in \mathrm{Spec}(L_+^{s_0}) \cap \mathrm{Spec}(L_-^{s_0})$, we have $\ker(N_{s_0}) = \mathrm{Span}\{\mathbf{u}_{s_0}^{(1)}, \mathbf{u}_{s_0}^{(2)}\}$, where the $\mathbf{u}_{s_0}^{(i)}$ are given in (3.21). Applying Proposition 4.2 with $\lambda_0 = 0$ and $n = 2$, we will show the following. Again, dot denotes $d/d\lambda$.

THEOREM 4.6. *Suppose $\dim \ker N_{s_0} = 2$, and denote the corresponding eigenfunctions of $L_+^{s_0}$ and $L_-^{s_0}$ by $u_{s_0}^{(1)}$ and $v_{s_0}^{(2)}$, respectively.*
  1. *If $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$, then $s^2\lambda \notin \mathrm{Spec}(N_s)$ for (λ, s) in a punctured neighborhood of $(0, s_0)$.*
  2. *If $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle = 0$ and*

$$(4.26) \qquad \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} + \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} \neq 0,$$

where $\widehat{u}_{s_0}^{(2)} \in \mathrm{dom}(L_+^{s_0})$ and $\widehat{v}_{s_0}^{(1)} \in \mathrm{dom}(L_-^{s_0})$ denote solutions to

$$(4.27) \qquad L_+^{s_0} \widehat{u}_{s_0}^{(2)} = v_{s_0}^{(2)}, \qquad -L_-^{s_0} \widehat{v}_{s_0}^{(1)} = u_{s_0}^{(1)},$$

then for $|\lambda| \ll 1$, there exist C^2 curves $s_1(\lambda)$ and $s_2(\lambda)$ such that
  (i) *$s_{1,2}^2(\lambda)\lambda \in \mathrm{Spec}\left(N_{s_{1,2}(\lambda)}\right)$,*
  (ii) *$s_{1,2}(0) = s_0$,*
  (iii) *$\dot{s}_{1,2}(0) = 0$,*
  and the concavities satisfy

$$(4.28) \qquad \ddot{s}_1(0) = -\frac{2s_0^5}{\ell} \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2}, \qquad \ddot{s}_2(0) = \frac{2s_0^5}{\ell} \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2}.$$

Moreover, there exist continuous curves $\mathbf{u}_{s_1(\lambda)}$ *and* $\mathbf{u}_{s_2(\lambda)}$ *of eigenfunctions such that*

$$(4.29) \qquad \mathbf{u}_{s_1(\lambda)} \to \mathbf{u}_{s_0}^{(1)} = \begin{pmatrix} u_{s_0}^{(1)} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_{s_2(\lambda)} \to \mathbf{u}_{s_0}^{(2)} = \begin{pmatrix} 0 \\ v_{s_0}^{(2)} \end{pmatrix}$$

as $\lambda \to 0$.

The condition (4.26) will be discussed in Remark 4.10 below.

Remark 4.7. As in Remark 3.17, the solutions $\widehat{u}_{s_0}^{(2)}$ and $\widehat{v}_{s_0}^{(1)}$ in (4.27) are not unique, but the expressions in (4.26) and (4.28) do not depend on the choice of solution.

We prove the theorem by studying the zero set of $m(\lambda, s) := \det M(\lambda, s)$, where M is given in (4.14). We thus start with some elementary calculations for the higher-order derivatives of m. These will be used to prove the existence of the eigenvalue curves $s_{1,2}(\lambda)$ and also to evaluate their first and second derivatives.

LEMMA 4.8. *Under the assumptions of Theorem 4.6, we have*

$$(4.30) \qquad m(0, s_0) = \frac{\partial m}{\partial s}(0, s_0) = \frac{\partial m}{\partial \lambda}(0, s_0) = \frac{\partial^2 m}{\partial s \partial \lambda}(0, s_0) = 0$$

and

$$(4.31)$$
$$\frac{\partial^2 m}{\partial s^2}(0, s_0) = -\frac{2\ell^2}{s_0^2}\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2 \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2, \qquad \frac{\partial^2 m}{\partial \lambda^2}(0, s_0) = -2s_0^4 \left\langle u_{s_0}^{(1)}, v_{s_0}^{(2)}\right\rangle^2.$$

Moreover, if $\left\langle u_{s_0}^{(1)}, v_{s_0}^{(2)}\right\rangle = 0$, *then*

$$(4.32) \quad \frac{\partial^3 m}{\partial s \partial \lambda^2}(0, s_0) = 2\ell s_0^3 \left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2 \left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)}\right\rangle - 2\ell s_0^3 \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2 \left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)}\right\rangle,$$

$$(4.33) \quad \frac{\partial^3 m}{\partial \lambda^3}(0, s_0) = 0, \qquad \frac{\partial^4 m}{\partial \lambda^4}(0, s_0) = 24 s_0^8 \left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)}\right\rangle \left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)}\right\rangle,$$

with $\widehat{u}_{s_0}^{(2)}$ *and* $\widehat{v}_{s_0}^{(1)}$ *as in* (4.27).

Proof. Writing $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $m = ad - bc$, we compute

$$\partial_s m = (\partial_s a) \, d + a \, (\partial_s d) - (\partial_s b) \, c - b \, (\partial_s c),$$
$$\partial_s^2 m = (\partial_s^2 a) \, d + 2 \, (\partial_s a) \, (\partial_s d) + a \, (\partial_s^2 d) - (\partial_s^2 b) \, c - 2 \, (\partial_s b) \, (\partial_s c) - b \, (\partial_s^2 c),$$

and so at $(0, s_0)$, we have

$$(4.34a) \qquad \partial_s m = 0, \qquad \partial_s^2 m = 2 \, (\partial_s a) \, (\partial_s d) - 2 \, (\partial_s b) \, (\partial_s c)$$

because $a = b = c = d = 0$ there (recall that $M(\lambda_0, s_0) = 0$). Similarly, we find that

$$(4.34b) \qquad \partial_\lambda m = 0,$$
$$(4.34c) \qquad \partial_\lambda^2 m = 2(\partial_\lambda a)(\partial_\lambda d) - 2(\partial_\lambda b)(\partial_\lambda c),$$
$$(4.34d) \qquad \partial_{s\lambda} m = (\partial_s a)(\partial_\lambda d) + (\partial_\lambda a)(\partial_s d) - (\partial_s b)(\partial_\lambda c) - (\partial_\lambda b)(\partial_s c)$$

at $(0, s_0)$. To evaluate the second derivatives, it remains to differentiate the components of M. By Proposition 4.2, for $i, j = 1, 2$, we have

$$(4.35) \qquad \frac{\partial M_{ij}}{\partial \lambda}(0, s_0) = -s_0^2 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \qquad \frac{\partial M_{ij}}{\partial s}(0, s_0) = \left\langle \partial_s B_{s_0} \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle.$$

It follows from (4.18) and (3.34) that

$$\frac{\partial M}{\partial \lambda}(0, s_0) = -s_0^2 \begin{pmatrix} 0 & \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \\ \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle & 0 \end{pmatrix},$$

so that at $(0, s_0)$, we have $\partial_\lambda a = \partial_\lambda d = 0$ and $\partial_\lambda b = \partial_\lambda c = -s_0^2 \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle$. Similarly, it follows from (4.18) and (3.29) that

$$(4.36) \qquad \frac{\partial M}{\partial s}(0, s_0) = \frac{\ell}{s_0} \begin{pmatrix} -\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2 & 0 \\ 0 & \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2 \end{pmatrix};$$

hence, at $(0, s_0)$, we have $\partial_s a = -s_0^{-1} \ell \left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2$, $\partial_s d = s_0^{-1} \ell \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2$, and $\partial_s b = \partial_s c = 0$. The claimed formulas for $\partial_s^2 m$, $\partial_{s\lambda} m$, and $\partial_\lambda^2 m$ now follow from (4.34).

If $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle = 0$, then $\partial_\lambda b = \partial_\lambda c = 0$ at $(0, s_0)$. This implies that $\partial_\lambda^3 m = 0$ and

$$(4.37) \qquad \partial_\lambda^4 m = 6 \left((\partial_\lambda^2 a)(\partial_\lambda^2 d) - (\partial_\lambda^2 b)(\partial_\lambda^2 c)\right), \quad \partial_{s\lambda\lambda} m = (\partial_s a)(\partial_\lambda^2 d) + (\partial_\lambda^2 a)(\partial_s d)$$

at $(0, s_0)$. Using (4.18) and (3.39), we obtain

$$(4.38) \qquad \frac{\partial^2 M}{\partial \lambda^2}(0, s_0) = -2s_0^4 \begin{pmatrix} \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle & 0 \\ 0 & \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle \end{pmatrix};$$

hence, $\partial_\lambda^2 b = \partial_\lambda^2 c = 0$, and it follows that

$$\partial_\lambda^4 m = 6(\partial_\lambda^2 a)(\partial_\lambda^2 d) = 24 s_0^8 \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle.$$

The claimed formula for $\partial_{s\lambda\lambda} m$ follows directly from (4.37). $\square$

The next elementary lemma will be used to prove differentiability of the eigenvalue curves in the second part of Theorem 4.6. In what follows, dot denotes $d/d\lambda$.

LEMMA 4.9. *If Δ is a smooth function with $\Delta(\lambda) = \alpha \lambda^4 + O(\lambda^5)$ as $|\lambda| \to 0$ for some $\alpha > 0$, then $\delta(\lambda) := \sqrt{\Delta(\lambda)}$ is C^2 near $\lambda = 0$, with $\dot{\delta}(0) = 0$ and $\ddot{\delta}(0) = 2\sqrt{\alpha}$.*

Proof. It is clear that δ is smooth except possibly at $\lambda = 0$. For the first derivative, we note that $\delta(\lambda)/\lambda \to 0$ as $\lambda \to 0$, so $\dot{\delta}(0) = 0$. For $\lambda \neq 0$, we compute

$$\dot{\delta}(\lambda) = \frac{1}{2} \Delta(\lambda)^{-1/2} \dot{\Delta}(\lambda).$$

Using $\Delta(\lambda) = \alpha \lambda^4 + O(\lambda^5)$ and $\dot{\Delta}(\lambda) = 4\alpha \lambda^3 + O(\lambda^4)$, we see that $\dot{\delta}(\lambda) \to 0$ as $\lambda \to 0$ and conclude that δ is C^1. Next, we observe that

$$\frac{\dot{\delta}(\lambda) - \dot{\delta}(0)}{\lambda} = \frac{1}{2} \frac{\lambda^2}{\sqrt{\Delta(\lambda)}} \frac{\dot{\Delta}(\lambda)}{\lambda^3} \to 2\sqrt{\alpha},$$

and hence $\ddot{\delta}(0)$ exists. A similar argument gives

$$\ddot{\delta}(\lambda) = -\frac{1}{4} \frac{\dot{\Delta}(\lambda)^2}{\Delta(\lambda)^{3/2}} + \frac{1}{2} \frac{\ddot{\Delta}(\lambda)}{\sqrt{\Delta(\lambda)}} \to 2\sqrt{\alpha}$$

as $\lambda \to 0$, so δ is C^2. $\square$

Proof of Theorem 4.6. By assumption, we have $m(0, s_0) = 0$. If $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$, Lemma 4.8 implies m has a strict local maximum at $(0, s_0)$, so m is negative (and in particular nonzero) in a punctured neighborhood of $(0, s_0)$. This proves the first case.

For the second case, we use the Malgrange preparation theorem (see [31, section IV.2]). We know from Lemma 4.8 that $m(0, s_0) = \partial_s m(0, s_0) = 0$ and $\partial_s^2 m(0, s_0) < 0$, so we can write

$$(4.39) \qquad m(\lambda, s) = Q(\lambda, s)P(\lambda, s)$$

in a neighborhood of $(0, s_0)$, where

$$(4.40) \qquad P(\lambda, s) = (s - s_0)^2 + B(\lambda)(s - s_0) + C(\lambda);$$

Q, B, and C are smooth, real-valued functions; and Q does not vanish in a neighborhood of $(0, s_0)$. This means m locally has the same zero set as P.

We claim that the discriminant $\Delta(\lambda) = B^2(\lambda) - 4C(\lambda)$ satisfies

$$(4.41) \qquad \Delta(\lambda) = \alpha\lambda^4 + O(\lambda^5) \text{ as } |\lambda| \to 0, \qquad \alpha = \frac{\ddot{B}(0)^2}{4} - \frac{C^{(4)}(0)}{6} > 0.$$

To see this, we compute the Taylor expansion of $\Delta(\lambda) = B(\lambda)^2 - 4C(\lambda)$ about $\lambda = 0$ and show that $\Delta(0) = \dot{\Delta}(0) = \ddot{\Delta}(0) = \dddot{\Delta}(0) = 0$. For this, it suffices to show that $B(0) = \dot{B}(0) = C(0) = \dot{C}(0) = \ddot{C}(0) = \dddot{C}(0) = 0$. That $\Delta^{(4)}(0) = 4!\alpha$ follows from the definition of $\Delta(\lambda)$.

Using Lemma 4.8, we obtain

$$m(0, s_0) = Q(0, s_0)C(0) = 0.$$

Since $Q(0, s_0) \neq 0$, this implies $C(0) = 0$. Similarly, we find that

$$\begin{aligned} \partial_\lambda m(0, s_0) &= Q(0, s_0)\dot{C}(0) = 0, \\ \partial_\lambda^2 m(0, s_0) &= Q(0, s_0)\ddot{C}(0) = 0, \\ \partial_\lambda^3 m(0, s_0) &= Q(0, s_0)\dddot{C}(0) = 0, \\ \partial_\lambda^4 m(0, s_0) &= Q(0, s_0)C^{(4)}(0) \end{aligned}$$

and

$$\begin{aligned} \partial_s m(0, s_0) &= Q(0, s_0)B(0) = 0, \\ \partial_{s\lambda} m(0, s_0) &= Q(0, s_0)\dot{B}(0) = 0, \\ \partial_{s\lambda\lambda} m(0, s_0) &= Q(0, s_0)\ddot{B}(0), \end{aligned}$$

which gives

$$B(0) = \dot{B}(0) = C(0) = \dot{C}(0) = \ddot{C}(0) = \dddot{C}(0) = 0.$$

We now observe that

$$\partial_s^2 m(0, s_0) = Q(0, s_0)\, \partial_s^2 P(0, s_0) = 2Q(0, s_0).$$

Using the first formula from (4.31), this implies that

$$(4.42) \qquad Q(0, s_0) = -\frac{\ell^2}{s_0^2} \left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2 \left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2.$$

Therefore, using (4.33),

$$(4.43) \qquad C^{(4)}(0) = \frac{\partial_\lambda^4 m(0, s_0)}{Q(0, s_0)} = -24 \frac{s_0^{10}}{\ell^2} \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2 \left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2}.$$

We similarly use (4.32) to compute

$$(4.44) \qquad \ddot{B}(0) = \frac{\partial_{s\lambda\lambda} m(0, s_0)}{Q(0, s_0)} = \frac{2 s_0^5}{\ell} \left\{ \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2} - \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2} \right\}.$$

Therefore,

$$(4.45) \qquad \alpha = \frac{\ddot{B}(0)^2}{4} - \frac{C^{(4)}(0)}{6} = \frac{s_0^{10}}{\ell^2} \left(\frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2} + \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2} \right)^2 > 0$$

because of (4.26), thus proving the claim.

Given (4.41), we have $\Delta(\lambda) > 0$ for small nonzero λ, and so the equation $P(\lambda, s) = 0$ has two solutions in s:

$$(4.46) \qquad s_\pm(\lambda) := \frac{-B(\lambda) \pm \sqrt{\Delta(\lambda)}}{2} + s_0.$$

It then follows from Lemma 4.9 that both $s_\pm(\lambda)$ are C^2 in a neighborhood of $\lambda = 0$, with $\dot{s}_\pm(0) = -\dot{B}(0)/2 = 0$ and

$$(4.47) \qquad \ddot{s}_\pm(0) = \frac{-\ddot{B}(0) \pm 2\sqrt{\alpha}}{2},$$

so the curves $s_\pm(\lambda)$ satisfy properties (i)–(iii) in the theorem. Substituting (4.44) and (4.45) into (4.47), we obtain

$$(4.48) \quad \ddot{s}_\pm(0) = \frac{s_0^5}{\ell} \left\{ \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2} - \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2} \pm \left| \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2} + \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2} \right| \right\}.$$

If the quantity inside the absolute value (which is nonzero by (4.26)) is positive, we get

$$(4.49) \qquad \ddot{s}_+(0) = \frac{2 s_0^5}{\ell} \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2}, \qquad \ddot{s}_-(0) = -\frac{2 s_0^5}{\ell} \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2},$$

in which case we define $s_1 := s_-$ and $s_2 := s_+$. If it is negative, we get

$$(4.50) \qquad \ddot{s}_-(0) = \frac{2 s_0^5}{\ell} \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell) \right)^2}, \qquad \ddot{s}_+(0) = -\frac{2 s_0^5}{\ell} \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell) \right)^2},$$

and we define $s_1 := s_+$ and $s_2 := s_-$.

To prove the existence of a continuous family of eigenfunctions, we define $M_1(\lambda) = M(\lambda, s_1(\lambda))$. If $\left(t_1(\lambda), t_2(\lambda) \right)^\top \in \ker M_1(\lambda)$ is nonzero, we know from (4.11) that

$$\mathbf{u}_{s_1(\lambda)} = \left(I + A(\lambda, s_1(\lambda)) \right) \left(t_1(\lambda) \mathbf{u}_{s_0}^{(1)} + t_2(\lambda) \mathbf{u}_{s_0}^{(2)} \right)$$

is an eigenfunction of $N_{s_1(\lambda)}$ for the eigenvalue $s_1^2(\lambda)\lambda$. We therefore need to understand the kernel of $M_1(\lambda)$.

By construction, we have $M_1(0) = 0$. Since $(\partial_\lambda M)(0, s_0) = 0$ and $\dot{s}_1(0) = 0$, we find that $\dot{M}_1(0) = 0$ and $\ddot{M}_1(0) = (\partial_\lambda^2 M)(0, s_0) + (\partial_s M)(0, s_0)\ddot{s}_1(0)$. Using (4.28), (4.36), and (4.38), we get

$$(4.51) \qquad \ddot{M}_1(0) = -2s_0^4 \big(\partial_x v_{s_0}^{(2)}(\ell)\big)^2 \left(\frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\big(\partial_x u_{s_0}^{(1)}(\ell)\big)^2} + \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\big(\partial_x v_{s_0}^{(2)}(\ell)\big)^2} \right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which is nonzero by (4.26). Writing $M_1(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$, it follows that $d(\lambda) \neq 0$ for small, nonzero values of λ, and so we can choose

$$\begin{pmatrix} t_1(\lambda) \\ t_2(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ -c(\lambda)/d(\lambda) \end{pmatrix} \in \ker M_1(\lambda)$$

for $\lambda \neq 0$. Since $c(0) = \dot{c}(0) = \ddot{c}(0) = d(0) = \dot{d}(0) = 0$ but $\ddot{d}(0) \neq 0$, we get $c(\lambda)/d(\lambda) \to 0$ as $\lambda \to 0$, and so

$$\lim_{\lambda \to 0} \big(I + A(\lambda, s_1(\lambda))\big) \big(t_1(\lambda)\mathbf{u}_{s_0}^{(1)} + t_2(\lambda)\mathbf{u}_{s_0}^{(2)}\big) = \mathbf{u}_{s_0}^{(1)}$$

as claimed. The result for $\mathbf{u}_{s_2(\lambda)}$ is proved in the same way.   □

Remark 4.10. The condition (4.26) implies $\Delta(\lambda) > 0$ for small nonzero λ and hence guarantees the existence of $s_\pm(\lambda)$. It also guarantees that $\ddot{s}_+(0) \neq \ddot{s}_-(0)$, as can be seen from (4.48). If (4.26) fails, then $\alpha = 0$, and we cannot use the result of Lemma 4.9. In this (nongeneric) case, one may compute higher derivatives of m in order to determine higher-order coefficients in the Taylor expansion of $\Delta(\lambda)$, but we do not pursue this here.

The following examples illustrate the two scenarios detailed in Theorem 4.6.

Example 4.11. The conditions in case (1) of Theorem 4.6 are satisfied if we take $L_+^s = L_-^s$, in which case $u_{s_0}^{(1)} = v_{s_0}^{(2)}$ at any crossing $(0, s_0)$, so that $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$. Each isolated crossing $(\lambda, s) = (0, s_0)$ is a consequence of a pair of purely imaginary eigenvalues passing through the origin as s increases. For clarity, in Figure 4, we
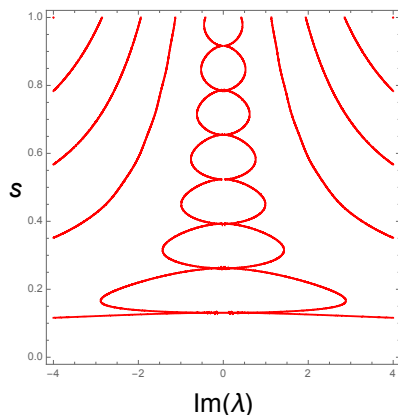

FIG. 4. *Imaginary eigenvalue curves* $s^2\lambda \in \mathrm{Spec}(N_s) \cap i\mathbb{R}$, *where* $L_-^s = L_+^s = -\partial_{xx} - 4s^2$ *and* $\ell = 12$. *Viewed from the* ηs-*plane, where* $\eta = \mathrm{Re}(\lambda)$, *a series of isolated crossings appear at* $\eta = 0$ *as* s *increases from* 0 *to* 1.
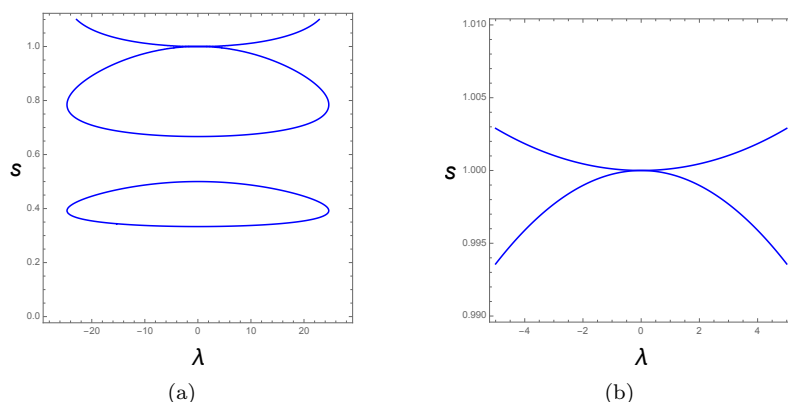
FIG. 5. (a) *Real eigenvalue curves* $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$, *where* $L_-^s = -\partial_{xx} - 4\pi^2 s^2$, $L_+^s = -\partial_{xx} - 9\pi^2 s^2$, *and* $\ell = 1$ *and* (b) *a blowup of the conjugate point* $(\lambda, s) = (0, 1)$.

have plotted the imaginary eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap i\mathbb{R}$ for the case when $L_-^s = L_+^s = -\partial_{xx} - 4s^2$ and $\ell = 12$ (here $\lambda \in \mathbb{C}$).

Example 4.12. Let $L = -\partial_{xx} + V(x)$ with domain (2.5), and define $L_\pm = L - \lambda_\pm$, where $\lambda_\pm \in \mathrm{Spec}(L)$ are distinct eigenvalues with eigenfunctions u_1 and v_2, so that $L_+ u_1 = L_- v_2 = 0$. Since $L_\pm$ is self-adjoint and $\lambda_+ \neq \lambda_-$, we have $\langle u_1, v_2 \rangle = 0$, and the conditions of case (2) in Theorem 4.6 are satisfied. (Recall the notation of (3.19) when $s_0 = 1$.)

The equations $L_+ \widehat{u}_2 = v_2$ and $-L_- \widehat{v}_1 = u_1$ are solved by $\widehat{u}_2 = \frac{1}{\lambda_- - \lambda_+} v_2$ and $\widehat{v}_1 = \frac{1}{\lambda_- - \lambda_+} u_1$, and it follows that

$$\int_0^\ell \widehat{u}_2 \, v_2 \, dx = \frac{1}{\lambda_- - \lambda_+} \int_0^\ell v_2^2 \, dx \quad \text{and} \quad \int_0^\ell \widehat{v}_1 \, u_1 \, dx = \frac{1}{\lambda_- - \lambda_+} \int_0^\ell u_1^2 \, dx$$

are nonzero and have the same sign. According to (4.28), this means the curves $s_{1,2}(\lambda)$ passing through $(0, 1)$ will have opposite concavity. This is illustrated in Figure 5, where we have plotted the real eigenvalue curves for a domain of length $\ell = 1$, choosing $L = -\partial_{xx}$, $\lambda_+ = 9\pi^2$, and $\lambda_- = 4\pi^2$.

4.4. The Maslov index at the nonregular corner. We are now in a position to calculate the corner term $\mathfrak{c}$ appearing in Theorem 2.2 (and defined in Definition 3.14) using the tools developed in subsections 4.2 and 4.3.

Since a nonregular crossing occurs at the initial point of Γ_3, we cannot use (3.4) to compute the Maslov index. We therefore take advantage of homotopy invariance, deforming the corner of the Maslov box to a path that only has simple regular crossings.

The index can then be deduced from the local behavior of the eigenvalue curves through $(0, 1)$ (see Theorems 2.9 and 4.6), which we quantify as follows. Given the curve $s(\lambda)$ from Theorem 2.9, there is an interval $(0, \widehat{\lambda})$ on which either $s(\lambda) > 1$ or $s(\lambda) < 1$ since the set $\{\lambda : s(\lambda) = 1\}$ is discrete; cf. Remark 3.15. Therefore, the quantity

(4.52)
$$s^\sharp(0) := \lim_{\lambda \to 0^+} \mathrm{sign}\,\big(s(\lambda) - 1\big) \in \{\pm 1\}$$

is well-defined. In the case that $s = s(\lambda)$ is analytic, $s^\sharp(0)$ is the sign of the first nonzero Taylor coefficient at $\lambda = 0$.

Remark 4.13. Recall from Theorem 2.9 that $\dot{s}(0) = 0$. Therefore, in the generic case where $\ddot{s}(0) \neq 0$, we simply have

$$(4.53) \qquad s^\sharp(0) = \operatorname{sign} \ddot{s}(0).$$

That is, the VK-type integrals in Theorem 2.9 determine $s^\sharp(0)$ (and hence the index $\mathfrak{c}$) provided the integrals are nonzero. However, it is important to note that the dichotomy $s^\sharp(0) = \pm 1$ holds even if $\ddot{s}(0) = 0$.

The same considerations apply to the curves $s_{1,2}(\lambda)$ from Theorem 4.6 (for which $\dot{s}_{1,2}(0) = 0$), so we define $s^\sharp_{1,2}(0)$ analogously and emphasize that in the generic case $\ddot{s}_{1,2}(0) \neq 0$, we have

$$(4.54) \qquad s^\sharp_{1,2}(0) = \operatorname{sign} \ddot{s}_{1,2}(0).$$

With this notation in place, we are ready to calculate $\mathfrak{c}$.

THEOREM 4.14. *The corner term $\mathfrak{c}$ from Definition 3.14 is calculated as follows:*
(1) *Suppose* $\dim \ker(N) = 1$, *and let* $s = s(\lambda)$ *be the eigenvalue curve through* $(0, 1)$.
  (i) *If* $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$, *then*

$$\mathfrak{c} = \frac{1}{2}(s^\sharp(0) - 1).$$

   That is, $\mathfrak{c} = 0$ *if* $s^\sharp(0) = +1$ *and* $\mathfrak{c} = -1$ *if* $s^\sharp(0) = -1$.
  (ii) *If* $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$, *then*

$$\mathfrak{c} = \frac{1}{2}(1 - s^\sharp(0)).$$

   That is, $\mathfrak{c} = 0$ *if* $s^\sharp(0) = +1$ *and* $\mathfrak{c} = +1$ *if* $s^\sharp(0) = -1$.
(2) *Suppose* $\dim \ker(N) = 2$, *with* $\ker(L_+) = \operatorname{Span}\{u_1\}$ *and* $\ker(L_-) = \operatorname{Span}\{v_2\}$. *If* $\langle u_1, v_2 \rangle \neq 0$, *then* $\mathfrak{c} = 0$. *If* $\langle u_1, v_2 \rangle = 0$ *and the condition* (4.26) *holds, we denote by* $s_{1,2}(\lambda)$ *the eigenvalue curves passing through* $(0, 1)$, *as in Theorem* 4.6. *Then*

$$(4.55) \qquad \mathfrak{c} = \frac{1}{2}(s^\sharp_1(0) - s^\sharp_2(0)).$$

We remark that formula (4.55) is simply the sum of the formulas for $\mathfrak{c}$ in cases (i) and (ii) of the simple case, identifying s with s_1 if $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ and s with s_2 if $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$. It is perhaps interesting to note that in (4.55), we have $\mathfrak{c} \in \{-1, 0, 1\}$, so that $\mathfrak{c}$ can never be $+2$ or -2 despite its being the contribution to the Maslov index from a two-dimensional crossing in this case.

Proof. We use a homotopy argument, deforming the top left corner of the Maslov box as shown in Figure 6.

We first consider the case $\dim \ker(N) = 1$. If $s^\sharp(0) > 0$, then the deformed path does not intersect $\mathcal{D}$, so we have $\mathfrak{c} = 0$. On the other hand, if $s^\sharp(0) < 0$, there will be a crossing at some point $(\lambda_*, s_*) = (\lambda_*, s(\lambda_*))$, with $0 < \lambda_* \ll 1$. This segment of the deformed path is parameterized by increasing s, so the relevant crossing form is

$$(4.56) \qquad \mathfrak{m}_{s_*}(q) = \frac{1}{s_*} \big\langle \big(\partial_s B_{s_*} - 2 s_* \lambda_*\big) \mathbf{u}_{s_*}, S \mathbf{u}_{s_*} \big\rangle,$$
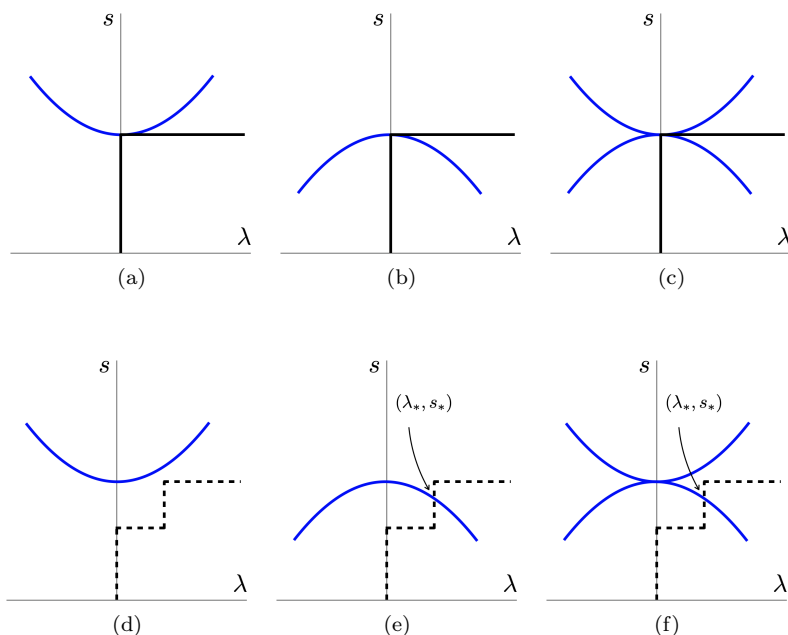
FIG. 6. *Neighborhood of the crossing* $(\lambda_0, s_0) = (0, 1)$ *featuring the eigenvalue curves (parabolas in blue) and the portion of the Maslov box passing through the corner* $(0, 1)$ *(in black) when* (a) $\dim \ker(N) = 1$ *and* $s^\sharp(0) > 0$, (b) $\dim \ker(N) = 1$ *and* $s^\sharp(0) < 0$, *and* (c) $\dim \ker(N) = 2$ *and* $s_1^\sharp(0) s_2^\sharp(0) < 0$. *The path (dashed) to which we homotope the top left corner of the Maslov box in* (a), (b), *and* (c) *is given in* (d), (e), *and* (f), *respectively.*

where $q = \mathrm{Tr}_{s_*} \mathbf{u}_{s_*}$. From Theorem 4.5, we obtain a continuous family of eigenfunctions with $\mathbf{u}_{s(\lambda)} \to \mathbf{u}$ as $\lambda \to 0$, so we can use Lemma 3.9 to compute

$$\lim_{\lambda \to 0} \frac{1}{s(\lambda)} \big\langle \big(\partial_s B_{s(\lambda)} - 2s(\lambda)\lambda\big) \mathbf{u}_{s(\lambda)}, S\mathbf{u}_{s(\lambda)} \big\rangle = \big\langle \partial_s B_1 \mathbf{u}_1, S\mathbf{u}_1 \big\rangle = \ell \left[-\big(u_1'(\ell)\big)^2 + \big(v_1'(\ell)\big)^2 \right].$$

By continuity, this has the same sign as the crossing form (4.56) at (λ_*, s_*), so we conclude that $\mathfrak{c} = -1$ if $0 \in \mathrm{Spec}(L_+)$ and $\mathfrak{c} = 1$ if $0 \in \mathrm{Spec}(L_-)$.

The argument for the case $\dim \ker(N) = 2$ is similar. Depending on the values of $s_1^\sharp(0)$ and $s_2^\sharp(0)$, there will be zero, one, or two crossings that contribute to the index $\mathfrak{c}$. These are necessarily simple crossings since $s_1(\lambda) \neq s_2(\lambda)$ for $\lambda \neq 0$ (see Remark 4.10). Moreover, if either $s_1^\sharp(0)$ or $s_2^\sharp(0)$ is positive, it does not contribute to the index.

Suppose $s_1^\sharp(0) < 0$, so that there is a crossing at some point $(\lambda_*, s_*) = (\lambda_*, s_1(\lambda_*))$. As in the first case, we need to compute the crossing form

$$\mathfrak{m}_{s_*}(q) = \frac{1}{s_*} \big\langle \big(\partial_s B_{s_*} - 2s_*\lambda_*\big) \mathbf{u}_{s_*}, S\mathbf{u}_{s_*} \big\rangle.$$

We use Theorem 4.6 to get

$$\lim_{\lambda \to 0} \frac{1}{s_1(\lambda)} \big\langle \big(\partial_s B_{s_1(\lambda)} - 2s_1(\lambda)\lambda\big) \mathbf{u}_{s_1(\lambda)}, S\mathbf{u}_{s_1(\lambda)} \big\rangle = \big\langle \partial_s B_1 \mathbf{u}_1^{(1)}, S\mathbf{u}_1^{(1)} \big\rangle$$

$$= -\ell \left(\partial_x u_1^{(1)}(\ell) \right)^2 < 0$$

and hence conclude that the crossing form at (λ_*, s_*) is negative. Similarly, if $s_2^\sharp(0) < 0$, there is a crossing at some point $(\lambda_*, s_2(\lambda_*))$ whose crossing form is positive because

$$\lim_{\lambda \to 0} \frac{1}{s_2(\lambda)} \langle (\partial_s B_{s_2(\lambda)} - 2s_2(\lambda)\lambda) \mathbf{u}_{s_2(\lambda)}, S\mathbf{u}_{s_2(\lambda)} \rangle = \langle \partial_s B_1 \mathbf{u}_1^{(2)}, S\mathbf{u}_1^{(2)} \rangle = \ell \left(\partial_x v_1^{(2)}(\ell) \right)^2 > 0.$$

In summary, the curve s_1 contributes 0 to $\mathfrak{c}$ if $s_1^\sharp(0) > 0$ and -1 if $s_1^\sharp(0) < 0$, whereas s_2 contributes 0 if $s_2^\sharp(0) > 0$ and 1 if $s_2^\sharp(0) < 0$. Adding these contributions completes the proof. ☐

We conclude this section by relating the concavity of the eigenvalue curves to the second-order Maslov crossing form.

PROPOSITION 4.15. *Assume the first-order crossing form* $\mathfrak{m}_{\lambda_0}$ *is identically zero at the crossing* $(\lambda_0, s_0) = (0, 1)$. *If the second-order crossing form* $\mathfrak{m}_{\lambda_0}^{(2)}$ *given in Lemma* 3.16 *is nondegenerate, then*

$$(4.57) \qquad \mathrm{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [0, \varepsilon]) = -n_-(\mathfrak{m}_{\lambda_0}^{(2)}).$$

Proof. We will prove this statement in the cases relevant to the current paper, that is, when $\dim \ker(N) = 1, 2$. Recall that nondegeneracy of $\mathfrak{m}_{\lambda_0}^{(2)}$ implies that $\ddot{s}(0) \neq 0$ if $\dim \ker(N) = 1$ and $\ddot{s}_{1,2}(0) \neq 0$ if $\dim \ker(N) = 2$. Therefore, (4.53) and (4.54) hold.

For the right-hand side of (4.57), if $\dim \ker(N) = 1$, Theorem 2.9 shows that the sign of $\ddot{s}(0)$ determines the sign of the VK-type integrals in (2.14) and (2.15) and therefore the sign of $\mathfrak{m}_{\lambda_0}^{(2)}$ given in (3.38). In particular, we observe the following:

(i) If $0 \in \mathrm{Spec}(L_+) \backslash \mathrm{Spec}(L_-)$, then $n_-(\mathfrak{m}_{\lambda_0}^{(2)}) = \begin{cases} 0 & \ddot{s}(0) > 0, \\ 1 & \ddot{s}(0) < 0. \end{cases}$

(ii) If $0 \in \mathrm{Spec}(L_-) \backslash \mathrm{Spec}(L_+)$, then $n_-(\mathfrak{m}_{\lambda_0}^{(2)}) = \begin{cases} 1 & \ddot{s}(0) > 0, \\ 0 & \ddot{s}(0) < 0. \end{cases}$

If $\dim \ker(N) = 2$, consider the matrix $\mathfrak{M}_{\lambda_0}^{(2)}$ of the second-order form $\mathfrak{m}_{\lambda_0}^{(2)}$, which is given in (3.39). Using (4.28), we see the following:

(iii) If $0 \in \mathrm{Spec}(L_+) \cap \mathrm{Spec}(L_-)$, then $n_-(\mathfrak{m}_{\lambda_0}^{(2)}) = \begin{cases} 0 & \ddot{s}_1(0) > 0, \ \ddot{s}_2(0) < 0, \\ 1 & \ddot{s}_1(0)\ddot{s}_2(0) > 0, \\ 2 & \ddot{s}_1(0) < 0, \ \ddot{s}_2(0) > 0. \end{cases}$

For the left-hand side of (4.57), let us define $\mathfrak{a} := \mathrm{Mas}(\Lambda(s, 0), \mathcal{D}; s \in [1-\varepsilon, 1])$ and $\mathfrak{b} := \mathrm{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [0, \varepsilon])$ and notice from (3.35) that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$. From the proof of Lemma 3.22, we know that the crossing form at $(0, 1)$ has $n_+(\mathfrak{m}_{s_0}) = \dim \ker(L_-)$, so Definition 3.1 gives $\mathfrak{a} = \dim \ker(L_-)$. Therefore,

$$(4.58) \qquad \mathfrak{b} = \mathfrak{c} - \dim \ker(L_-).$$

Using the values of $\mathfrak{c}$ computed in in Theorem 4.14, we confirm that $\mathfrak{b} = -n_-(\mathfrak{m}_{\lambda_0}^{(2)})$ in cases (i), (ii), and (iii) described above, as claimed. ☐

5. Applications. In this section, we give some applications of the theory of sections 3 and 4. We begin with the proof of Corollaries 2.7 and 2.8 and Theorem 2.11, which are consequences of Theorem 2.2 and Theorem 4.14. We then give formulas for the concavity of the NLS spectral curves and recover the classical VK criterion for a particular one-parameter family of stationary states. Finally, we relate our results to the Krein index theory.

5.1. The Jones–Grillakis instability theorem. We first prove the compact interval analogue of the Jones–Grillakis instability theorem, Corollary 2.7, and its consequence Corollary 2.8.

Proof of Corollary 2.7. From Theorem 2.2, we have $n_+(N) \geq 1$ provided $P - Q \neq \mathfrak{c}$. The result now follows from Theorem 4.14, which guarantees $\mathfrak{c} \in \{-1, 0\}$ when $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ and $\mathfrak{c} \in \{0, 1\}$ when $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$. $\square$

Proof of Corollary 2.8. We claim that $Q = 0$, $P \geq 1$ and $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ under the assumptions of the corollary. Once this has been shown, the result follows immediately from Corollary 2.7.

Since ϕ is nonconstant and satisfies Neumann boundary conditions, we have $0 \in \mathrm{Spec}(L_+)$, with eigenfunction ϕ'. Moreover, each stationary point of ϕ in the interior of its domain corresponds to a conjugate point for L_+: If $\phi'(x_0) = 0$ for some $x_0 \in (0, \ell)$, then $0 \in \mathrm{Spec}(L_+^{s_0})$ for $s_0 = x_0/\ell$, with eigenfunction $\phi(s_0 x)$. It then follows from Lemma 3.19 that $P \geq 1$.

We next consider L_-^s for $s \in (0, 1]$. Under Hypothesis 2.5, the general solution to the differential equation $L_-^s w = 0$ is

$$(5.1) \qquad w(x) = c_1 \phi(sx) + c_2 \phi(sx) \int_0^x \frac{1}{\phi(st)^2} \, dt,$$

where the second fundamental solution was obtained via the method of reduction of order and is well-defined since $\phi(x) \neq 0$ for all $x \in [0, \ell]$ implies $1/\phi^2$ is integrable. It follows that

$$(5.2) \qquad \phi(sx) \int_0^x \frac{1}{\phi(st)^2} \, dt \geq 0$$

for all $x \in [0, \ell]$, with equality when $x = 0$. Dirichlet boundary conditions on w then dictate that $c_1 = c_2 = 0$, and we conclude that $0 \notin \mathrm{Spec}(L_-^s)$ for all $s \in (0, 1]$. In particular, $0 \notin \mathrm{Spec}(L_-)$, and Lemma 3.19 implies $Q = 0$. $\square$

5.2. VK-type (in)stability criteria. For the proof Theorem 2.11, we will need two preliminary results. The first of these mimics [32, Corollary 1.1] and follows from the equivalent self-adjoint formulation of the eigenvalue problem (3.44); see Lemma 3.21.

LEMMA 5.1. *If $Q = 0$ or $P = 0$, then $\mathrm{Spec}(N_s) \subset \mathbb{R} \cup i\mathbb{R}$ for all $s \in (0, 1]$.*

Proof. Fix $s \in (0, 1]$. If $Q = 0$, then L_-^s is nonnegative by Lemma 3.20. By Lemma 3.21, the eigenvalue problem (3.44) is equivalent to (3.45). The operator $\left(L_-^s|_{X_c}\right)^{1/2} \Pi L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2}$ acting in X_c is self-adjoint, and therefore $s^4 \lambda^2 \in \mathbb{R}$. Then $s \in \mathbb{R}$ implies $\lambda \in \mathbb{R} \cup i\mathbb{R}$. The case $P = 0$ follows similarly. $\square$

We next prove that the Maslov index is monotone in λ if either $Q = 0$ or $P = 0$.

LEMMA 5.2. *If $Q = 0$, then the crossing form $\mathfrak{m}_{\lambda_0}$ is strictly positive for any crossing with $\lambda_0 > 0$ and $s_0 = 1$, while if $P = 0$, then $\mathfrak{m}_{\lambda_0}$ is strictly negative at all such crossings. Consequently,*

$$(5.3) \qquad n_+(N) = \begin{cases} \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) & \text{if } Q = 0, \\ -\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) & \text{if } P = 0. \end{cases}$$

(Recall that $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = \mathrm{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [\varepsilon, \lambda_\infty]).$)

Proof. Assume $\lambda_0 > 0$ with eigenfunction $\mathbf{u}_1 = (u_1, v_1)^\top$, so that (3.44) holds with $\lambda = \lambda_0$ and $s = 1$. Note that both u_1 and v_1 are necessarily nontrivial due to the coupling of the eigenvalue equations for $\lambda \neq 0$. If $Q = 0$, we apply $\langle \cdot, v_1 \rangle$ to the first equation of (3.44) to obtain

$$(5.4) \qquad \langle L_- v_1, v_1 \rangle = -\lambda_0 \langle u_1, v_1 \rangle = \frac{\lambda_0}{2} \mathfrak{m}_{\lambda_0}(q), \quad q = \operatorname{Tr}\mathbf{u}_1,$$

using formula (3.30). Now $0 \neq u_1 \in \operatorname{ran}(L_-)$ implies v_1 has a component lying in $\ker(L_-)^\perp$. Since $Q = 0$, it follows that $\langle L_- v_1, v_1 \rangle > 0$. Thus, $\mathfrak{m}_{\lambda_0}(q) > 0$ at all crossings along Γ_3^ε if $Q = 0$. If $P = 0$, one applies $\langle \cdot, u_1 \rangle$ to the second equation of (3.44) at $(\lambda_0, 1)$, and a similar argument yields that $\langle L_+ u_1, u_1 \rangle = -\frac{\lambda_0}{2}\mathfrak{m}_{\lambda_0}(q) > 0$. Thus, $\mathfrak{m}_{\lambda_0}(q) < 0$ at all crossings on Γ_3^ε if $P = 0$. $\qquad\square$

Proof of Theorem 2.11. Consider the eigenvalue curve $s = s(\lambda)$ through the point $(\lambda, s) = (0, 1)$, for which $\dot{s}(0) = 0$ as stated in Theorem 2.9.

We start with the case $P = 1, Q = 0$ and $0 \in \operatorname{Spec}(L_-) \backslash \operatorname{Spec}(L_+)$. If $\ddot{s}(0) > 0$, then by Theorem 4.14, we have $\mathfrak{c} = 0$. Since $Q = 0$, by Lemma 5.2 and (3.56), we have $n_+(N) = P - \mathfrak{c} = 1$. On the other hand, if $\ddot{s}(0) < 0$, then by Theorem 4.14, we have $\mathfrak{c} = 1$, and by the same argument, $n_+(N) = P - \mathfrak{c} = 0$. It then follows from Lemma 5.1 that $\operatorname{Spec}(N) \subset i\mathbb{R}$.

The case where $Q = 1, P = 0$ and $0 \in \operatorname{Spec}(L_+) \backslash \operatorname{Spec}(L_-)$ is similar. If $\ddot{s}(0) > 0$, then $\mathfrak{c} = 0$ by Theorem 4.14, and Lemma 5.2 and (3.56) imply $n_+(N) = Q + \mathfrak{c} = 1$. If $\ddot{s}(0) < 0$, then $\mathfrak{c} = -1$ by Theorem 4.14; hence, $n_+(N) = 0$. By Lemma 5.1, we deduce that $\operatorname{Spec}(N) \subset i\mathbb{R}$. $\qquad\square$

5.3. Concavity computations for NLS. Working under Hypothesis 2.5, in this subsection, we compute the sign of $\ddot{s}(0)$ via the VK-type integrals given in Theorem 2.9. In what follows, $s(\lambda)$ is the eigenvalue curve through $(\lambda_0, s_0) = (0, 1)$.

5.3.1. The L_+ integral. We first consider the case when L_+ has a nontrivial kernel. The following result allows us to compute $\ddot{s}(0)$ when ϕ satisfies Neumann boundary conditions.

PROPOSITION 5.3. *Assume Hypothesis* 2.5 *and that* $0 \in \operatorname{Spec}(L_+) \backslash \operatorname{Spec}(L_-)$ *with eigenfunction* ϕ'. *If* $\{p, q\}$ *is a fundamental set of solutions to the differential equation* $L_- v = 0$ *initialized at the identity, then* $q(\ell) \neq 0$ *and*

$$(5.5) \qquad \operatorname{sign} \ddot{s}(0) = \operatorname{sign}\left[\left(\int_0^\ell p^2 dx\right) - \frac{p(\ell)}{q(\ell)}\ell^2\right].$$

Proof. First, note that $\ker(N) = \operatorname{Span}\{(\phi', 0)^\top\}$. Now by case (2) of Theorem 2.9, we have

$$\operatorname{sign} \ddot{s}(0) = \operatorname{sign} \int_0^\ell \widehat{v}\,\phi'\,dx,$$

where $\widehat{v}$ is the unique solution to the inhomogeneous boundary value problem

$$(5.6) \qquad L_- \widehat{v} = \phi', \qquad \widehat{v}(0) = \widehat{v}(\ell) = 0.$$

Let $\{p, q\}$ be a fundamental set of solutions to the homogeneous equation $L_- \widehat{v} = 0$ such that

$$(5.7) \qquad \begin{pmatrix} p(0) & q(0) \\ p'(0) & q'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\phi(0) \neq 0$, the first solution is given by $p(x) = \phi(x)/\phi(0)$. We have $p'(\ell) = 0, p(\ell) \neq 0$, while $q(\ell) \neq 0$ since $q(0) = 0$ and $0 \notin \mathrm{Spec}(L_-)$. By Abel's identity,

$$(5.8) \qquad p(x)q'(x) - q(x)p'(x) = 1 \quad \forall \ x \in [0, \ell].$$

The general solution to the differential equation $L_- \widehat{v} = \phi'$ is thus

$$(5.9) \qquad \widehat{v}(x) = Ap(x) + Bq(x) - \frac{x\phi(x)}{2},$$

where it is easily verified that $-x\phi(x)/2$ is a particular solution. Imposing the boundary conditions on $\widehat{v}$ to determine the constants A and B, we find that the unique solution to (5.6) is

$$\widehat{v}(x) = \frac{1}{2}\left(\frac{\ell\phi(\ell)}{q(\ell)}q(x) - x\phi(x)\right).$$

It remains to compute sign $\int_0^\ell \widehat{v}\phi' dx$. Since $\phi(x) = p(x)\phi(0)$, we have

$$\int_0^\ell \widehat{v}(x)\phi'(x)dx = \int_0^\ell \frac{1}{2}\left(\frac{\ell\phi(\ell)}{q(\ell)}q(x) - x\phi(x)\right)p'(x)\phi(0)\,dx$$
$$= \frac{\phi(0)^2\ell p(\ell)}{2q(\ell)}\int_0^\ell q(x)p'(x)dx - \frac{\phi(0)^2}{2}\int_0^\ell xp(x)p'(x)dx.$$

For the second integral, we obtain

$$\int_0^\ell xp(x)p'(x)dx = \frac{1}{2}\left(\ell p(\ell)^2 - \int_0^\ell p(x)^2 dx\right),$$

while for the first, we integrate by parts and appeal to (5.8) to arrive at

$$\int_0^\ell q(x)p'(x)dx = \frac{1}{2}\left(q(\ell)p(\ell) - \ell\right).$$

Therefore,

$$\int_0^\ell \widehat{v}(x)\phi'(x)dx = \frac{\phi(0)^2\ell p(\ell)}{4q(\ell)}\left(q(\ell)p(\ell) - \ell\right) - \frac{\phi(0)^2}{4}\left(\ell p(\ell)^2 - \int_0^\ell p(x)^2 dx\right)$$
$$= \frac{\phi(0)^2}{4}\left(\int_0^\ell p(x)^2 dx - \frac{p(\ell)}{q(\ell)}\ell^2\right),$$

and (5.5) follows. $\qquad \square$

Remark 5.4. If ϕ is nonvanishing, the second solution q can be determined using reduction of order; see (5.10) and also the proof of Corollary 2.8. When ϕ has zeros, the second solution is given by the Rofe–Beketov formula [72, Lemma 2]; however, the resulting expression is significantly more complicated and does not appear to be useful for our analysis.

The following result serves as an application of Proposition 5.3 in the case when the stationary state is either strictly positive or strictly negative over its domain.

COROLLARY 5.5. *Under the assumptions of Proposition* 5.3, *for nonconstant solutions to* (2.9) *satisfying $\phi(x) \neq 0$ for all $x \in [0, \ell]$, we have $\ddot{s}(0) > 0$.*

Proof. In the case when ϕ has no zeros on the interval $[0, \ell]$, the method of reduction of order allows us to write

$$(5.10) \qquad q(x) = p(x) \int_0^x \frac{1}{p(t)^2} dt,$$

where the nonvanishing of p ensures $1/p^2$ is integrable. This gives

$$\int_0^\ell p(x)^2 dx - \frac{p(\ell)}{q(\ell)} \ell^2 = \frac{\left(\int_0^\ell \frac{1}{p^2} dx \right) \left(\int_0^\ell p^2 dx \right) - \ell^2}{\left(\int_0^\ell \frac{1}{p^2} dx \right)},$$

and so

$$(5.11) \qquad \operatorname{sign} \ddot{s}(0) = \operatorname{sign} \left[\left(\int_0^\ell \frac{1}{p^2} dx \right) \left(\int_0^\ell p^2 dx \right) - \ell^2 \right].$$

By virtue of the Cauchy–Schwarz inequality,

$$\ell = \int_0^\ell p(x) \frac{1}{p(x)} \, dx \le \sqrt{\int_0^\ell p^2(x) dx} \sqrt{\int_0^\ell \frac{1}{p(x)^2} dx},$$

where we have equality only when p and $1/p$ are linearly dependent, that is, when ϕ is constant. Since we have assumed a nonconstant solution, the inequality is strict, and we conclude that (5.11) is positive. $\qquad \square$

Remark 5.6.   The statement of Corollary 5.5 may also be proven using Remark 2.10 since $L_- > 0$ for stationary states that are nonvanishing over $[0, \ell]$ (as was shown in the proof of Corollary 2.8). However, the proof given above is a nice illustration of Proposition 5.3, a more general result that holds for any nonconstant ϕ.

5.3.2. The L_- integral: Recovering classical VK. We now consider the case when L_- has a nontrivial kernel (spanned by ϕ). We show that the associated VK-type integral in (2.14) of Theorem 2.9 recovers a compact interval analogue of the classical VK integral expression

$$(5.12) \qquad \frac{\partial}{\partial \beta} \int_{-\infty}^\infty \phi^2 \, dx$$

associated with a stationary state $\phi \in L^2(\mathbb{R})$ solving (2.9) (see [66, Theorem 4.4, page 215]). The key observation is that $\partial_\beta \phi(\cdot; \beta)$ solves the differential equation $L_+ \widehat{u} = \phi$ associated with case (1) of Theorem 2.9, and this naturally leads to the expressions (5.14) and (5.15), which clearly resemble (5.12). This is not true for the equation $L_- \widehat{v} = \phi'$ associated with case (2) of Theorem 2.9, for which a recovery of a compact interval analogue of (5.12) is thus not possible. In what follows, $\phi'(x; \beta)$ refers to $\frac{d\phi}{dx}(x; \beta)$, while the β derivative will be denoted by ∂_β.

PROPOSITION 5.7. *Assume Hypothesis* 2.5, *and let ϕ_0 be a solution to* (2.9) *with parameter β_0 that satisfies $\phi_0(0) = \phi_0(\ell) = 0$. There exists a unique one-parameter family of solutions $\beta \mapsto \hat{\phi}(\cdot; \beta)$ to* (2.9), *defined in a neighborhood of β_0, such that*

$$(5.13) \qquad \hat{\phi}(0; \beta) = \hat{\phi}(\ell; \beta) = 0$$

for all β near β_0 and $\hat{\phi}(\cdot; \beta_0) = \phi_0$. In terms of this family, the VK-type integral in (2.14) *is*

$$(5.14) \qquad \int_0^\ell \widehat{u}\, v\, dx = \frac{1}{2} \frac{\partial}{\partial \beta}\bigg|_{\beta=\beta_0} \int_0^\ell \hat{\phi}(x; \beta)^2\, dx.$$

More generally, if $\beta \mapsto \phi(\cdot; \beta)$ is any C^1 family of solutions to (2.9) *satisfying $\phi(\cdot; \beta_0) = \phi_0$, then the integral in* (2.14) *can be written as*

$$(5.15)$$
$$\int_0^\ell \widehat{u}\, v\, dx = \frac{1}{2} \frac{\partial}{\partial \beta}\bigg|_{\beta=\beta_0} \int_0^\ell \phi(x; \beta)^2\, dx$$
$$+ \left((-1)^Q \partial_\beta \phi(0; \beta_0) + \partial_\beta \phi(\ell; \beta_0) \right) \left(\frac{\partial_\beta \phi(0; \beta_0) + (-1)^Q \partial_\beta \phi(\ell; \beta_0)}{q(\ell)} + \partial_\beta \phi'(\ell; \beta_0) \right).$$

Furthermore, if $P = 1$, $Q = 0$, and (5.14) *or* (5.15) *is positive (resp., negative), then the standing wave $\widehat{\psi}(x, t) = e^{i\beta_0 t}\phi_0(x)$ is spectrally unstable (resp., spectrally stable).*

Proof. The existence of ϕ_0 implies that the associated operators

$$L_- = -\partial_{xx} - f(\phi_0^2) - \beta_0,$$
$$L_+ = -\partial_{xx} - 2f'(\phi_0^2)\phi_0^2 - f(\phi_0^2) - \beta_0$$

have $\phi_0 \in \ker(L_-)$ and hence $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$. Consider the function

$$(5.16) \quad F : \left(H^2(0,\ell) \cap H_0^1(0,\ell) \right) \times \mathbb{R} \longrightarrow L^2(0,\ell), \qquad F(\phi, \beta) = \phi'' + f(\phi^2)\phi + \beta\phi,$$

in terms of which (2.9) and (5.13) become $F(\phi, \beta) = 0$. It can be shown that F is continuously Fréchet differentiable (see [14, section 2.2]), with

$$(5.17) \qquad\qquad DF(\phi_0, \beta_0)(u, \gamma) = \gamma\phi_0 - L_+ u.$$

Since $0 \notin \mathrm{Spec}(L_+)$, this implies $DF(\phi_0, \beta_0)(\cdot, 0) = -L_+$ is invertible, so the implicit function theorem guarantees the existence of a C^1 function

$$(5.18) \qquad\qquad (\beta_0 - \epsilon, \beta_0 + \epsilon) \to H^2(0,\ell) \cap H_0^1(0,\ell), \quad \beta \mapsto \hat{\phi}(\cdot; \beta),$$

such that $F(\hat{\phi}(\cdot; \beta), \beta) = 0$ for all $|\beta - \beta_0| < \epsilon$.

Turning to the integral in (2.14), where now $v = \phi_0$, we need to solve

$$(5.19) \qquad\qquad L_+ \widehat{u} = \phi_0, \qquad \widehat{u}(0) = \widehat{u}(\ell) = 0.$$

Using the family constructed above, which is C^1 in β, we differentiate (2.9) with respect to β and evaluate at β_0 to obtain

$$(5.20) \qquad\qquad L_+ \partial_\beta \hat{\phi}(x; \beta_0) = \phi_0(x).$$

Now differentiating (5.13) (which holds for *all* β near β_0) with respect to β and evaluating at β_0 yields

$$(5.21) \qquad\qquad \partial_\beta \hat{\phi}(0; \beta_0) = \partial_\beta \hat{\phi}(\ell; \beta_0) = 0.$$

Therefore, $\widehat{u}(x) = \partial_\beta \hat{\phi}(x; \beta_0)$ is the *unique* solution to (5.19), and substituting this into the VK-type integral in (2.14) with $v = \phi_0$ yields (5.14).

Now let $\beta \mapsto \phi(\cdot; \beta)$ be an arbitrary family of solutions to (2.9) (again for β close to β_0) such that $\phi(x; \beta_0) = \phi_0(x)$. To solve (5.19), note that (5.20) still holds for the family $\phi(\cdot; \beta_0)$, and thus the general solution to $L_+ \widehat{u} = \phi_0$ is

$$(5.22) \qquad \widehat{u}(x) = A p(x) + B q(x) + \partial_\beta \phi(x; \beta_0),$$

where $\{p, q\}$ is now a fundamental set of solutions to the homogeneous equation $L_+ \widehat{u} = 0$ satisfying (5.7). Since $\phi'(0; \beta_0) \neq 0$, we may set $p(x) = \phi'(x; \beta_0)/\phi'(0; \beta_0)$. A brief look at the Hamiltonian for (2.9) indicates that intersections of any fixed orbit with $\phi = 0$ are symmetric about $\phi' = 0$; from this, along with the Sturm–Liouville theory applied to $\phi(\cdot; \beta_0) = \phi_0 \in \ker(L_-)$, we deduce that we necessarily have $\phi'(\ell; \beta_0) = (-1)^{Q+1} \phi'(0; \beta_0)$ and therefore that $p(\ell) = (-1)^{Q+1}$. Evaluating (2.9) at $x = \ell$, we also find that $\phi''(\ell; \beta_0) = 0$; hence, $p'(\ell) = 0$. Thus,

$$(5.23) \qquad \begin{pmatrix} p(\ell) & q(\ell) \\ p'(\ell) & q'(\ell) \end{pmatrix} = \begin{pmatrix} (-1)^{Q+1} & * \\ 0 & (-1)^{Q+1} \end{pmatrix},$$

where $q'(\ell) = (-1)^{Q+1}$ because (5.23) must have unit determinant by virtue of Abel's identity (see (5.8)). In addition, $q(\ell) \neq 0$ since $0 \notin \mathrm{Spec}(L_+)$ and $q(0) = 0$.

Imposing the boundary conditions $\widehat{u}(0) = \widehat{u}(\ell) = 0$ and using (5.23) allows us to determine the constants A and B. We find that the unique solution to (5.19) is

$$(5.24) \quad \widehat{u}(x) = -\partial_\beta \phi(0; \beta_0)\, p(x) + \frac{(-1)^{Q+1} \partial_\beta \phi(0; \beta_0) - \partial_\beta \phi(\ell; \beta_0)}{q(\ell)}\, q(x) + \partial_\beta \phi(x; \beta_0).$$

Multiplying (5.24) by ϕ_0 and integrating the first two terms by parts yields (5.15). The statement regarding spectral stability follows immediately from Theorem 2.11. $\square$

Remark 5.8. The one-parameter family constructed abstractly in (5.18) via the implicit function theorem leads to the simplest expression for the VK-type integral on a compact interval. However, this is only useful in practice if one can determine this family explicitly, which may not be possible. For this reason, we have included formula (5.15), which holds for *any* one-parameter family of solutions to the standing wave equation that starts at ϕ_0.

Remark 5.9. When the spatial domain is the entire real line, it is known that for power-law nonlinearities of the form $f(\phi^2) = \phi^{2p}$, $p > 0$, strictly positive localized stationary states (for which $\beta < 0$, $P = 1$, and $Q = 0$) are spectrally stable[1] for $p \leq 2$ and spectrally unstable for $p > 2$ (see [66, Corollary 4.3, page 216]). The result follows from a change in sign of the VK integral (5.12) (see [66, Theorem 4.4, page 215]). Moving to the compact interval, we investigated whether an analogous phenomenon holds for stationary states ϕ_0 that likewise satisfy $\beta < 0$, $P = 1$, and $Q = 0$. We found that our numerical experiments were in line with the result on the real line when $p = 1, 2$, for which we found no spectrally unstable waves. Interestingly, however, for $p \in (2, p_0), p_0 \approx 5$, we observed the existence of a β-dependent threshold value of the interval length $\ell = \ell^*$ separating spectral stability ($\ell < \ell^*$) and spectral instability ($\ell > \ell^*$). This agrees with the instability result on the real line (for these values of p) in the sense that we recover it (numerically) on taking $\ell \to +\infty$. Theorem 2.11 indicates that this change in stability at $\ell = \ell^*$ should be reflected in a change in concavity of the eigenvalue curve passing through $(\lambda, s) = (0, 1)$, and indeed we observe

---

[1]The critical case $p = 2$ is spectrally stable but nonlinearly unstable due to algebraically growing solutions of the linearized system; see [66, Remark 4.3, page 217].
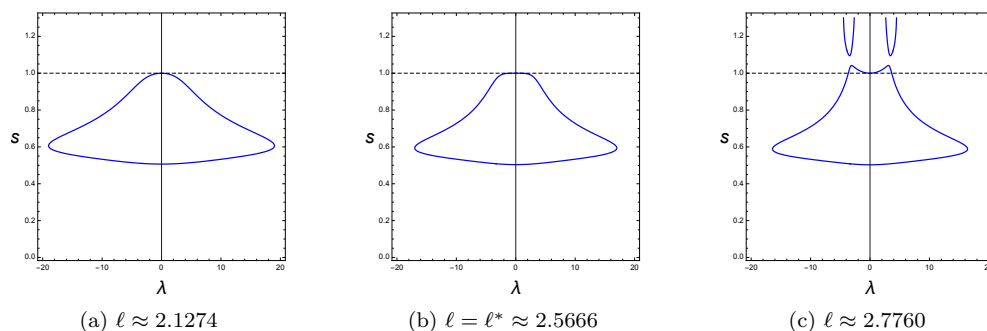
FIG. 7. *Eigenvalue curves* $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$ *under Hypothesis* 2.5(i) *for* T-*periodic stationary states* ϕ_0 *satisfying* $\phi_0(0) = \phi_0(\ell) = 0$, *with nonlinearity* $f(\phi^2) = \phi^6$, $\beta = -2$, *and domain length* $\ell = T/2$ *indicated. These* ϕ_0 *correspond to orbits located outside the homoclinic orbit and in the right half plane of Figure* 1(a). *(Note the phase plane for* (2.9) *with* $f(\phi^2) = \phi^6$ *is qualitatively similar to Figure* 1(a).*) Eigenvalues of* N *are given by intersections with the dashed line at* $s = 1$. *At* $\ell = \ell^*$, *we computed* $\ddot{s}(0) \approx 0$ *to four decimal places.*

this numerically. Figure 7 displays the real eigenvalue curves for three T-periodic stationary states ϕ_0 satisfying the Dirichlet boundary conditions $\phi_0(0) = \phi_0(\ell) = 0$, $\ell = T/2$, for differing ℓ. The sign of $\ddot{s}(0)$ at $(\lambda, s) = (0, 1)$ switches from negative to positive as ℓ increases through $\ell = \ell^*$. By Theorem 2.11, the underlying standing wave becomes unstable, which is confirmed by the emergence of a positive real eigenvalue in Figure 7(c).

Remark 5.10. In the previous example, note that at the critical value $\ell = \ell^*$, we have $\dim \ker(N) = 1$ and $\ddot{s}(0) = 0$. This corresponds to the nongeneric case in Remark 4.13, where $s^{\sharp}(0) \neq \mathrm{sign}\, \ddot{s}(0)$, and the second-order crossing form $\mathfrak{m}_{\lambda_0}^{(2)}$ in Lemma 3.16 is degenerate. A brief calculation using the Fredholm alternative indicates that the algebraic multiplicity of $\lambda = 0 \in \mathrm{Spec}(N)$ is at least four.

5.4. Connections with existing eigenvalue counts. We now give a comparison of our lower bound (2.7) with the one given in [51, equation (3.9)] (see (5.33) below); see also [53, Theorem 7.1.16]. We will show that the contribution to the Maslov index from the nonregular crossing (see Definition 3.14) is equal to the difference in negative indices of matrices arising in constrained eigenvalue counts for $L_{\pm}$. We refer the reader to [20] for an alternate approach to the constrained eigenvalue problem using the Maslov index. Throughout this section, $\{\mathbf{u}_1, \ldots \mathbf{u}_n\}$ is a basis for $\ker(N)$ with $n \leq 2$. We assume that the crossing $(\lambda_0, s_0) = (0, 1)$ is nonregular in the λ direction, with first-order crossing form $\mathfrak{m}_{\lambda_0}$ in (3.30) that is identically zero. We further assume that the second-order crossing form $\mathfrak{m}_{\lambda_0}^{(2)}$ in (3.36) is nondegenerate. The notation $n_-(A)$ refers to the number of negative eigenvalues of the self-adjoint operator or symmetric matrix A. Recall then that $P = n_-(L_+)$ and $Q = n_-(L_-)$.

Define the diagonal, self-adjoint operator

$$(5.25) \qquad L := \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \qquad \mathrm{dom}(L) := \mathrm{dom}(N),$$

so that $N = JL$. The eigenvalue problem (2.1) may then be written as

$$(5.26) \qquad JL\mathbf{u} = \lambda \mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{u}(\ell) = 0.$$

We denote the generalized eigenvectors of $N = JL$ by $\widehat{\mathbf{v}}_i$, i.e.,

$$(5.27) \qquad JL\widehat{\mathbf{v}}_i = \mathbf{u}_i, \quad JL\mathbf{u}_i = 0, \qquad i = 1, \ldots, n.$$

As in Remark 3.17, the Fredholm alternative and the fact that $\mathfrak{m}_{\lambda_0} = 0$ guarantee the existence of solutions to the first n equations in (5.27), so the algebraic multiplicity of $\lambda = 0$ is at least $2n$. Nondegeneracy of $\mathfrak{M}_{\lambda_0}^{(2)}$ then implies the algebraic multiplicity is exactly $2n$.

The matrix D in [51, equation (3.1)] is the $n \times n$ matrix with entries

$$(5.28) \qquad D_{ij} = \langle \widehat{\mathbf{v}}_i, L\widehat{\mathbf{v}}_j \rangle = -\langle \widehat{\mathbf{v}}_i, J\mathbf{u}_j \rangle,$$

where the second equality follows since $JL\widehat{\mathbf{v}}_i = \mathbf{u}_i$ implies $L\widehat{\mathbf{v}}_i = J^{-1}\mathbf{u}_i = -J\mathbf{u}_i$. It is used to determine the number of negative eigenvalues of L restricted to $\operatorname{ran} JL = [\ker(JL)^*]^\perp$ (see [51, Theorem 3.1]). Denoting $\dim \ker L_\pm = z_\pm \in \{0, 1\}$ so that $z_+ + z_- = n$, notice that the off-diagonal structure of JL implies that its eigenvectors and generalized eigenvectors may be written as

$$(5.29) \qquad \mathbf{u}_i = \begin{cases} (u_i, 0)^\top, \\ (0, v_i)^\top, \end{cases} \qquad \widehat{\mathbf{v}}_i = \begin{cases} (0, \widehat{v}_i)^\top, & i = 1, \ldots, z_+, \\ (\widehat{u}_i, 0)^\top, & i = z_+ + 1, \ldots, n, \end{cases}$$

where, by (5.27), the functions $u_i, v_i, \widehat{u}_i, \widehat{v}_i$ satisfy

$$-L_-\widehat{v}_i = u_i, \quad L_+ u_i = 0, \qquad i = 1, \ldots, z_+,$$
$$L_+\widehat{u}_i = v_i, \quad L_- v_i = 0, \qquad i = z_+ + 1, \ldots, n.$$

The matrix D thus has the block form (as in [51, section 3.3])

$$D = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix},$$

where

$$(5.30) \qquad \begin{aligned} [D_-]_{ij} &= \langle \widehat{v}_i, L_-\widehat{v}_j \rangle = -\langle \widehat{v}_i, u_j \rangle, & i, j = 1, \ldots, z_+, \\ [D_+]_{ij} &= \langle \widehat{u}_{z_++i}, L_+\widehat{u}_{z_++j} \rangle = \langle \widehat{u}_{z_++i}, v_{z_++j} \rangle, & i, j = 1, \ldots, z_-. \end{aligned}$$

The matrices D_+ and D_- are themselves used in constrained eigenvalue counts. Namely, if D_+ and D_- are nondegenerate, then

$$(5.31) \qquad n_-(\Pi L_+ \Pi) = P - n_-(D_+), \qquad n_-(\Pi L_- \Pi) = Q - n_-(D_-),$$

where Π is the orthogonal projection onto $[\ker(L_-) \oplus \ker(L_+)]^\perp$ (see [51, Lemma 3.1]).

Now noticing that the entries of $\mathfrak{M}_{\lambda_0}^{(2)}$ are given by

$$\left[\mathfrak{M}_{\lambda_0}^{(2)}\right]_{ij} = -2\langle \widehat{\mathbf{v}}_i, S\mathbf{u}_j \rangle = \begin{cases} -2\langle \widehat{v}_i, u_j \rangle, & i, j = 1, \ldots, z_+, \\ -2\langle \widehat{u}_i, v_j \rangle, & i, j = z_+ + 1, \ldots, n, \\ 0 & \text{elsewhere}, \end{cases}$$

because of (3.37) and (5.29), we are led to the observation that

$$(5.32) \qquad \mathfrak{M}_{\lambda_0}^{(2)} = 2 \begin{pmatrix} D_- & 0 \\ 0 & -D_+ \end{pmatrix}.$$

Clearly, $\mathfrak{M}_{\lambda_0}^{(2)}$ is nonsingular if and only if D_+ and D_- are nonsingular. Under this condition, in the notation of the current paper, equation (3.9) from [51] reads

$$(5.33) \qquad n_+(N) \geq |n_-(\Pi L_+ \Pi) - n_-(\Pi L_- \Pi)| = |P - Q - n_-(D_+) + n_-(D_-)|.$$

Comparing (5.33) with (2.7), we might naively expect that $\mathfrak{c} = n_-(D_+) - n_-(D_-)$. We confirm this in the following proposition.

PROPOSITION 5.11. *If $n \leq 2$ and $\mathfrak{M}_{\lambda_0}^{(2)}$ is nondegenerate, then*

$$\mathfrak{c} = n_-(D_+) - n_-(D_-). \tag{5.34}$$

That is, the contribution to the Maslov index from the crossing $(\lambda, s) = (0, 1)$ is precisely the difference of the "correction factors" counting the mismatch in negative dimensions between $L_\pm$ and their constrained counterparts (see (5.31)).

Proof. Recall the definition of $\mathfrak{b}$ given in the proof of Proposition 4.15. By the same proposition, if $n \leq 2$, we have

$$\mathfrak{b} = -n_-(\mathfrak{M}_{\lambda_0}^{(2)}) = -\big(n_-(D_-) + n_-(-D_+)\big), \tag{5.35}$$

where the last equality follows from (5.32). Notice that D_+ is a $z_- \times z_-$ matrix. Since D_+ is nondegenerate, it follows that

$$n_-(-D_+) = z_- - n_-(D_+). \tag{5.36}$$

Thus, by (5.35),

$$\mathfrak{b} = -n_-(D_-) - (\dim \ker L_- - n_-(D_+)), \tag{5.37}$$

and using (4.58) and rearranging gives (5.34).   □

A direct relationship between the matrices $D_\pm$ and the concavities of the eigenvalue curves follows from Theorem 2.9, Lemma 3.16, Theorem 4.6, and (5.32). In particular, it is straightforward to show that the following:

(i) If $0 \in \mathrm{Spec}(L_-) \backslash \mathrm{Spec}(L_+)$, then $z_+ = 0$ and

$$\mathrm{sign}\ \mathfrak{m}_{\lambda_0}^{(2)}(q) = -\mathrm{sign}\ D_+ = -\mathrm{sign}\ \ddot{s}(0). \tag{5.38a}$$

(ii) If $0 \in \mathrm{Spec}(L_+) \backslash \mathrm{Spec}(L_-)$, then $z_- = 0$ and

$$\mathrm{sign}\ \mathfrak{m}_{\lambda_0}^{(2)}(q) = \mathrm{sign}\ D_- = \mathrm{sign}\ \ddot{s}(0). \tag{5.38b}$$

(iii) If $0 \in \mathrm{Spec}(L_-) \cap \mathrm{Spec}(L_+)$, then $z_- = z_+ = 1$ and

$$\mathrm{sign}\ \ddot{s}_1(0) = \mathrm{sign}\ D_-, \qquad \mathrm{sign}\ \ddot{s}_2(0) = \mathrm{sign}\ D_+ \tag{5.38c}$$

(provided (4.26) holds so that $\mathrm{sign}\ \ddot{s}_1(0) = -\mathrm{sign}\ \langle \widehat{v}_1, u_1 \rangle$ and $\mathrm{sign}\ \ddot{s}_2(0) = \mathrm{sign}\ \langle \widehat{u}_2, v_2 \rangle$).

We finish the present work with an application of our results to a formula relating the number of eigenvalues of JL that are either unstable or susceptible to instability-inducing bifurcations, to the negative index of the constrained operator $L|_{X_c}$, $X_c := \mathrm{ran}(JL)$, known as the *Hamiltonian–Krein index theorem* (see [53, Theorem 7.1.5] or [60, Theorem 2.3]). For the eigenvalue problem (2.1)–(2.3), because L is diagonal and the symplectic matrix J is invertible, this formula reduces to that in [51, Theorem 3.3], which in the notation of the current paper reads

$$k_r + 2k_c + 2k_i^- = P + Q - n_-(D_-) - n_-(D_+). \tag{5.39}$$

Here, $k_r := n_+(N)$, k_c is the number eigenvalues lying in the open first quadrant, and k_i^- is the number of eigenvalues on the positive imaginary axis with negative Krein signature (see [51]). Note that (5.39) holds provided D_+ and D_- are nonsingular (and since P, Q, and n are finite, where $\dim \ker(JL) = \frac{1}{2} \dim \mathrm{gker}(JL) = n$; see [53, section 7.1.3] or [51] for details). In light of our earlier results, this leads to the following.

PROPOSITION 5.12. *Equation* (5.39) *may be written in one of the following equivalent forms:*

$$(5.40) \qquad k_r + 2k_c + 2k_i^- = -Mas(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) + 2P - 2n_-(D_+),$$
$$(5.41) \qquad = Mas(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) + 2Q - 2n_-(D_-).$$

Proof. Using Proposition 5.11 and Lemma 3.22, we can rearrange (5.39) to read

$$(5.42) \qquad k_r + 2k_c + 2k_i^- = \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) + \mathfrak{c} + 2P - 2n_-(D_+).$$

Then (5.40) follows from (5.42) using (3.55). A similar manipulation yields

$$(5.43) \qquad k_r + 2k_c + 2k_i^- = -\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) - \mathfrak{c} + 2Q - 2n_-(D_-),$$

in which case (5.41) follows from (5.43) via (3.55). $\qquad\square$

COROLLARY 5.13. *If $P = 0$ or $Q = 0$, then $k_c = k_i^- = 0$.*

Proof. If $P = 0$, then by Lemma 5.2, we have $k_r = n_+(N) = -Mas(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$. Furthermore, if $P = 0$, then L_+ is a nonnegative operator in $L^2(0, \ell)$, and in particular, $n_-(D_+) = 0$. Canceling terms on both sides of (5.40), we get

$$(5.44) \qquad 2k_c + 2k_i^- = 0,$$

as required. Note we could have argued that $k_c = 0$ using Lemma 5.1. The case $Q = 0$ is similar: $k_r = n_+(N) = \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$ by Lemma 5.2, and we have $L_- \geq 0$ in $L^2(0, \ell)$. Thus, $n_-(D_-) = 0$, and (5.41) yields the result. $\qquad\square$

In the case that $L_\pm$ are invertible, the previous result agrees with that given in [45, Corollary 2.26], where the dimension of intersecting cones is zero because $P = 0$ or $Q = 0$. The result for $Q = 0$ is a special case of the formula in [51, Remark 3.1, equation (3.10)].

COROLLARY 5.14. *If either $k_r = 0$ or the Maslov index of the path $\lambda \to \Lambda(\lambda, 1), \lambda \in [\varepsilon, \lambda_\infty], 0 < \varepsilon \ll 1$ is monotone in λ, then $k_c + k_i^- = Q - n_-(D_-) = P - n_-(D_+)$.*

Proof. If $k_r = 0$, the statement follows from (5.40) and (5.41) on noticing that $k_r = n_+(N) = 0$ implies $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = 0$ by (3.57).

Monotonicity of the Lagrangian path stated means that the crossing form (3.30) has the same sign at all crossings along Γ_3. In this case, $k_r = n_+(N) = \pm\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$, and the statement follows from (5.40) or (5.41). $\qquad\square$

Remark 5.15. Monotonicity in λ is guaranteed if $P = 0$ or $Q = 0$. However, the Maslov index is in general not monotone when $P, Q \geq 1$, and attempts to compute the terms k_c and k_i^- in these cases using the formulas above have so far been limited.

We finish with a numerical example to illustrate the scenario in Corollary 5.14. In Figure 8, we have plotted the *complex* eigenvalue curves for $s \in (0, 1]$ under Hypothesis 2.5(i), associated with a Jacobi cnoidal function ϕ_0 (see Figure 1(a)) satisfying $\phi_0'(0) = \phi_0'(\ell) = 0$. Precisely, the blue curves represent real eigenvalues, the red curves represent imaginary eigenvalues, and the purple curves represent eigenvalues lying off the real and imaginary axes. It was computed that the minimum point of each blue connected component (for which $\lambda = 0$) corresponds to a point of nontrivial kernel for L_+^s, while the maximum point of each such component corresponds to a point of nontrivial kernel for L_-^s. Note that by a simple rescaling, we can apply the formulas
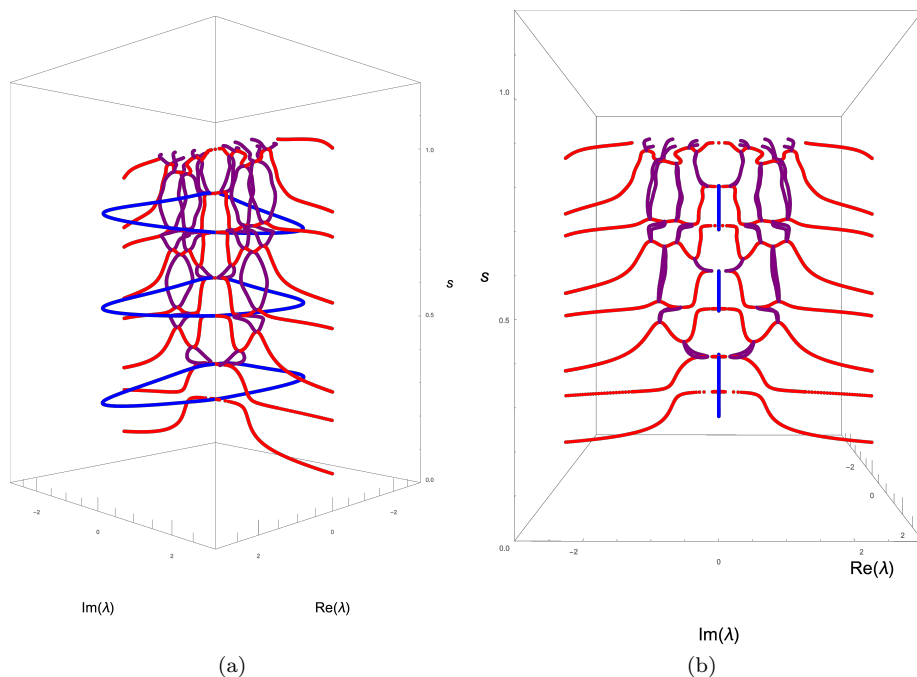
(a)   (b)

FIG. 8. *Real (blue), imaginary (red), and complex (purple) eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{C}$, $\lambda \in [-3,3] \times [-3i, 3i] \subset \mathbb{C}$, $s \in (0,1]$, under Hypothesis 2.5(i) for a T-periodic stationary state ϕ_0 with $f(\phi^2) = \phi^2$ satisfying $\phi_0'(0) = \phi_0'(\ell) = 0$, where $\ell = 2T = 13.3854$. Here, ϕ_0 is a Jacobi cnoidal function corresponding to an orbit located outside the homoclinic orbit in Figure 1(a). Figures (a) and (b) give two different viewpoints of the same curves. The eigenvalues were computed using Mathematica's `NDEigenvalues` command.*

of the current section to the rescaled operators $N_s, L_\pm^s$ for *any* $s \in (0,1]$. Consider then a horizontal plane at $s = s_* \approx 0.85$ in Figure 8, which coincides with the maximum point of the top blue connected component. By the above considerations and Lemma 3.19 applied to the interval $(0, s^*)$ instead of $(0,1)$, we have $P = n_-(L_+^{s_*}) = 3$ and $Q = n_-(L_+^{s_*}) = 2$. Since $0 \in \mathrm{Spec}(L_-^{s^*}) \backslash \mathrm{Spec}(L_+^{s^*})$, D_- is null (see (5.30)), and hence $n_-(D_-) = 0$. Figure 8 clearly shows $k_r = 0$ for $s = s^*$, and by Corollary 5.14, we deduce that $n_-(D_+) = 1$ and $k_c + k_i^- = 2$. (It was confirmed numerically that $k_c = 2$.) A similar analysis can be done for any of the minima or maxima of the blue connected components in Figure 8 or indeed for any horizontal plane which does not intersect the blue curves (for which $k_r = 0$).

REFERENCES

[1] V. I. ARNOL'D, *On a characteristic class entering into conditions of quantization*, Funktsional. Anal. i Prilozhen, 1 (1967), pp. 1–14.

[2] V. I. ARNOL'D, *Sturm theorems and symplectic geometry*, Funktsional. Anal. i Prilozhen., 19 (1985), pp. 1–10.

[3] T. BAIRD, P. CORNWELL, G. COX, C. K. R. T. JONES, AND R. MARANGELL, *Generalized Maslov indices for non-Hamiltonian systems*, SIAM J. Math. Anal., 54 (2022), pp. 1623–1668, https://doi.org/10.1137/20M1381319.

[4] M. BECK, G. COX, C. JONES, Y. LATUSHKIN, K. MCQUIGHAN, AND A. SUKHTAYEV, *Instability of pulses in gradient reaction-diffusion systems: A symplectic approach*, Philos. Trans. Roy. Soc. A, 376 (2018), 20170187, https://doi.org/10.1098/rsta.2017.0187.

[5] A. BOSE AND C. K. R. T. JONES, *Stability of the in-phase travelling wave solution in a pair of coupled nerve fibers*, Indiana Univ. Math. J., 44 (1995), pp. 189–220, https://doi.org/10.1512/iumj.1995.44.1984.

[6] R. BOTT, *On the iteration of closed geodesics and the Sturm intersection theory*, Comm. Pure Appl. Math., 9 (1956), pp. 171–206.

[7] N. BOTTMAN, B. DECONINCK, AND M. NIVALA, *Elliptic solutions of the defocusing NLS equation are stable*, J. Phys. A, 44 (2011), pp. 285201–285224, https://doi.org/10.1088/1751-8113/44/28/285201.

[8] S. CAPPELL, R. LEE, AND E. MILLER, *On the Maslov index*, Comm. Pure Appl. Math., 47 (1994), pp. 121–186.

[9] F. CHARDARD, F. DIAS, AND T. J. BRIDGES, *Computing the Maslov index of solitary waves. I. Hamiltonian systems on a four-dimensional phase space*, Phys. D, 238 (2009), pp. 1841–1867, https://doi.org/10.1016/j.physd.2009.05.008.

[10] F. CHARDARD, F. DIAS, AND T. J. BRIDGES, *On the Maslov index of multi-pulse homoclinic orbits*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 465 (2009), pp. 2897–2910, https://doi.org/10.1098/rspa.2009.0155.

[11] F. CHARDARD, F. DIAS, AND T. J. BRIDGES, *Computing the Maslov index of solitary waves, Part 2: Phase space with dimension greater than four*, Phys. D, 240 (2011), pp. 1334–1344, https://doi.org/10.1016/j.physd.2011.05.014.

[12] C.-N. CHEN AND X. HU, *Maslov index for homoclinic orbits of Hamiltonian systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), pp. 589–603, https://doi.org/10.1016/j.anihpc.2006.06.002.

[13] C.-N. CHEN AND X. HU, *Stability analysis for standing pulse solutions to FitzHugh-Nagumo equations*, Calc. Var. Partial Differential Equations, 49 (2014), pp. 827–845, https://doi.org/10.1007/s00526-013-0601-0.

[14] R. COLEMAN, *Calculus on Normed Vector Spaces*, Springer-Verlag, Berlin, 2012.

[15] P. CORNWELL, *Opening the Maslov box for traveling waves in skew-gradient systems: Counting eigenvalues and proving (in)stability*, Indiana Univ. Math. J., 68 (2019), pp. 1801–1832, https://doi.org/10.1512/iumj.2019.68.7831.

[16] P. CORNWELL AND C. K. R. T. JONES, *On the existence and stability of fast traveling waves in a doubly diffusive FitzHugh-Nagumo system*, SIAM J. Appl. Dyn. Syst., 17 (2018), pp. 754–787, https://doi.org/10.1137/17M1149432.

[17] P. CORNWELL AND C. K. R. T. JONES, *A stability index for travelling waves in activator-inhibitor systems*, Proc. Roy. Soc. Edinburgh Sect. A, 150 (2020), pp. 517–548, https://doi.org/10.1017/prm.2018.92.

[18] G. COX, C. K. R. T. JONES, Y. LATUSHKIN, AND A. SUKHTAYEV, *The Morse and Maslov indices for multidimensional Schrödinger operators with matrix-valued potentials*, Trans. Amer. Math. Soc., 368 (2016), pp. 8145–8207, https://doi.org/10.1090/tran/6801.

[19] G. COX, C. K. R. T. JONES, AND J. L. MARZUOLA, *A Morse index theorem for elliptic operators on bounded domains*, Comm. Partial Differential Equations, 40 (2015), pp. 1467–1497, https://doi.org/10.1080/03605302.2015.1025979.

[20] G. COX AND J. L. MARZUOLA, *A symplectic perspective on constrained eigenvalue problems*, J. Differential Equations, 266 (2019), pp. 2924–2952.

[21] B. DECONINCK AND B. L. SEGAL, *The stability spectrum for elliptic solutions to the focusing NLS equation*, Phys. D, 346 (2017), pp. 1–19, https://doi.org/10.1016/j.physd.2017.01.004.

[22] B. DECONINCK AND J. UPSAL, *The orbital stability of elliptic solutions of the focusing nonlinear Schrödinger equation*, SIAM J. Math. Anal., 52 (2020), pp. 1–41, https://doi.org/10.1137/19M1240757.

[23] J. DENG AND C. K. R. T. JONES, *Multi-dimensional Morse index theorems and a symplectic view of elliptic boundary value problems*, Trans. Amer. Math. Soc., 363 (2011), pp. 1487–1508, https://doi.org/10.1090/S0002-9947-2010-05129-3.

[24] J. J. DUISTERMAAT, *On the Morse index in variational calculus*, Adv. Math., 21 (1976), pp. 173–195, https://doi.org/10.1016/0001-8708(76)90074-8.

[25] H. M. EDWARDS, *A generalized Sturm theorem*, Ann. Math. (2), 80 (1964), pp. 22–57, https://doi.org/10.2307/1970490.

[26] T. GALLAY AND M. HĂRĂGUŞ, *Stability of small periodic waves for the nonlinear Schrödinger equation*, J. Differential Equations, 234 (2007), pp. 544–581, https://doi.org/10.1016/j.jde.2006.12.007.

[27] T. GALLAY AND M. HĂRĂGUS, *Orbital stability of periodic waves for the nonlinear Schrödinger equation*, J. Dynam. Differential Equations, 19 (2007), pp. 825–865, https://doi.org/10.1007/s10884-007-9071-4.

[28] T. GALLAY AND D. PELINOVSKY, *Orbital stability in the cubic defocusing NLS equation:* I. *Cnoidal periodic waves*, J. Differential Equations, 258 (2015), pp. 3607–3638, https://doi.org/10.1016/j.jde.2015.01.018.

[29] R. GIAMBO, P. PICCIONE, AND A. PORTALURI, *On the Maslov Index of Symplectic Paths That Are Not Transversal to the Maslov Cycle: Semi-Riemannian Index Theorems in the Degenerate Case*, preprint, arXiv:math/0306187, 2003.

[30] R. GIAMBÒ, P. PICCIONE, AND A. PORTALURI, *Computation of the Maslov index and the spectral flow via partial signatures*, C. R. Math. Acad. Sci. Paris, 338 (2004), pp. 397–402, https://doi.org/10.1016/j.crma.2004.01.004.

[31] M. GOLUBITSKY AND V. GUILLEMIN, *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics 14, Springer-Verlag, Berlin, 1973.

[32] M. GRILLAKIS, *Linearized instability for nonlinear Schrödinger and Klein-Gordon equations*, Comm. Pure Appl. Math., 41 (1988), pp. 747–774, https://doi.org/10.1002/cpa.3160410602.

[33] M. GRILLAKIS, J. SHATAH, AND W. STRAUSS, *Stability theory of solitary waves in the presence of symmetry,* I, J. Funct. Anal., 74 (1987), pp. 160–197, https://doi.org/10.1016/0022-1236(87)90044-9.

[34] M. GRILLAKIS, J. SHATAH, AND W. STRAUSS, *Stability theory of solitary waves in the presence of symmetry,* II, J. Funct. Anal., 94 (1990), pp. 308–348.

[35] P. GRINFELD, *Hadamard's formula inside and out*, J. Optim. Theory Appl., 146 (2010), pp. 654–690.

[36] J. HADAMARD, *Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées* (1908), in Œuvres de J. Hadamard, Vol. 2, C.N.R.S., Paris, 1968.

[37] A. HATCHER, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[38] D. HENRY, *Perturbation of the Boundary in Boundary-Value Problems of Partial Differential Equations*, London Mathematical Society Lecture Note Series 318, Cambridge University Press, Cambridge, 2005.

[39] P. HOWARD, *The Maslov index and spectral counts for linear Hamiltonian Systems on $\mathbb{R}$*, J. Dynam. Differential Equations, 33 (2021).

[40] P. HOWARD, S. JUNG, AND B. KWON, *The Maslov index and spectral counts for linear Hamiltonian systems on* $[0,1]$, J. Dynam. Differential Equations, 30 (2018), pp. 1703–1729, https://doi.org/10.1007/s10884-017-9625-z.

[41] P. HOWARD, Y. LATUSHKIN, AND A. SUKHTAYEV, *The Maslov index for Lagrangian pairs on* $\mathbb{R}^{2n}$, J. Math. Anal. Appl., 451 (2017), pp. 794–821.

[42] P. HOWARD, Y. LATUSHKIN, AND A. SUKHTAYEV, *The Maslov and Morse indices for system Schrödinger operators on* $\mathbb{R}$, Indiana Univ. Math. J., 67 (2018), pp. 1765–1815, https://doi.org/10.1512/iumj.2018.67.7462.

[43] P. HOWARD AND A. SUKHTAYEV, *The Maslov and Morse indices for Schrödinger operators on* $[0, 1]$, J. Differential Equations, 260 (2016), pp. 4499–4549.

[44] P. HOWARD AND A. SUKHTAYEV, *Renormalized oscillation theory for linear Hamiltonian systems on* $[0, 1]$ *via the Maslov index*, J. Dynam. Differential Equations, (2022).

[45] M. HĂRĂGUŞ AND T. KAPITULA, *On the spectra of periodic waves for infinite-dimensional Hamiltonian systems*, Phys. D, 237 (2008), pp. 2649–2671, https://doi.org/10.1016/j.physd.2008.03.050.

[46] T. IVEY AND S. LAFORTUNE, *Spectral stability analysis for periodic traveling wave solutions of NLS and CGL perturbations*, Phys. D, 237 (2008), pp. 1750–1772, https://doi.org/10.1016/j.physd.2008.01.017.

[47] R. K. JACKSON, R. MARANGELL, AND H. SUSANTO, *An instability criterion for nonlinear standing waves on nonzero backgrounds*, J. Nonlinear Sci., 24 (2014), pp. 1177–1196, https://doi.org/10.1007/s00332-014-9215-8.

[48] C. K. R. T. JONES, *Instability of standing waves for nonlinear Schrödinger-type equations*, Ergodic Theory Dynam. Systems, 8 (1988), pp. 119–138, https://doi.org/10.1017/S014338570000938X.

[49] C. K. R. T. JONES, Y. LATUSHKIN, AND R. MARANGELL, *The Morse and Maslov indices for matrix Hill's equations*, Proc. Sympos. Pure Math., 87 (2013), pp. 205–233.

[50] C. K. R. T. JONES, Y. LATUSHKIN, AND S. SUKHTAIEV, *Counting spectrum via the Maslov index for one dimensional θ-periodic Schrödinger operators*, Proc. Amer. Math. Soc., 145 (2017), pp. 363–377, https://doi.org/10.1090/proc/13192.

[51] T. KAPITULA, P. G. KEVREKIDIS, AND B. SANDSTEDE, *Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems*, Phys. D, 195 (2004), pp. 263–282, https://doi.org/10.1016/j.physd.2004.03.018.

[52] T. KAPITULA AND K. PROMISLOW, *Stability indices for constrained self-adjoint operators*, Proc. Amer. Math. Soc., 140 (2012), pp. 865–880, https://doi.org/10.1090/S0002-9939-2011-10943-2.

[53] T. KAPITULA AND K. PROMISLOW, *Spectral and Dynamical Stability of Nonlinear Waves*, Vol. 185, Springer-Verlag, Berlin, 2013.

[54] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980.

[55] R. KOLLÁR AND P. D. MILLER, *Graphical Krein signature theory and Evans-Krein functions*, SIAM Rev., 56 (2014), pp. 73–123, https://doi.org/10.1137/120891423.

[56] Y. LATUSHKIN AND A. SUKHTAIEV, *Hadamard-type formulas via the Maslov form*, J. Evol. Equ., 17 (2017), pp. 443–472, https://doi.org/10.1007/s00028-016-0354-2.

[57] Y. LATUSHKIN AND S. SUKHTAIEV, *The Maslov index and the spectra of second order elliptic operators*, Adv. Math., 329 (2018), pp. 422–486, https://doi.org/10.1016/j.aim.2018.02.027.

[58] Y. LATUSHKIN AND S. SUKHTAIEV, *First-Order Asymptotic Perturbation Theory for Extensions of Symmetric Operators*, preprint, arXiv:2012.00247, 2020.

[59] Y. LATUSHKIN AND S. SUKHTAIEV, *An index theorem for Schrödinger operators on metric graphs*, in Analytic Trends in Mathematical Physics, Contemporary Mathematics 741, American Mathematical Society, Providence, RI, 2020, pp. 105–119, https://doi.org/10.1090/conm/741/14922.

[60] Z. LIN AND C. ZENG, *Instability, Index Theorem, and Exponential Trichotomy for Linear Hamiltonian PDEs*, Memoirs of the American Mathematical Society 275, American Mathematical Society, Providence, RI, 2022, https://doi.org/10.1090/memo/1347.

[61] R. MARANGELL, C. K. R. T. JONES, AND H. SUSANTO, *Localized standing waves in inhomogeneous Schrödinger equations*, Nonlinearity, 23 (2010), pp. 2059–2080, https://doi.org/10.1088/0951-7715/23/9/002.

[62] R. MARANGELL, H. SUSANTO, AND C. K. R. T. JONES, *Unstable gap solitons in inhomogeneous nonlinear Schrödinger equations*, J. Differential Equations, 253 (2012), pp. 1191–1205, https://doi.org/10.1016/j.jde.2012.04.010.

[63] V. MASLOV, *Theory of Perturbations and Asymptotic Methods*, Izdat. Moskov. Gos. Univ., Moscow, 1965.

[64] J. MILNOR, *Morse Theory*, Annals of Mathematics Studies 51, Princeton University Press, Princeton, NJ, 1963.

[65] J. A. PAVA, *Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg-de Vries equations*, J. Differential Equations, 235 (2007), pp. 1–30.

[66] D. E. PELINOVSKY, *Localization in Periodic Potentials: From Schrödinger Operators to the Gross-Pitaevskii Equation*, Cambridge University Press, Cambridge, 2011.

[67] J. W. S. RAYLEIGH, *Theory of Sound*, Vol. 2, Dover, New York, 1945.

[68] F. RELLICH, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach Science Publishers, New York, 1969.

[69] J. ROBBIN AND D. SALAMON, *The Maslov index for paths*, Topology, 32 (1993), pp. 827–844.

[70] G. ROWLANDS, *On the stability of solutions of the non-linear Schrödinger equation*, IMA J. Appl. Math., 13 (1974), pp. 367–377.

[71] R. RUSIN, R. MARANGELL, AND H. SUSANTO, *Symmetry breaking bifurcations in the NLS equation with an asymmetric delta potential*, Nonlinear Dynam., 100 (2020), pp. 3815–3824.

[72] K. M. SCHMIDT, *Critical coupling constants and eigenvalue asymptotics of perturbed periodic Sturm-Liouville operators*, Comm. Math. Phys., 211 (2000), pp. 465–485, https://doi.org/10.1007/s002200050822, https://doi-org.qe2a-proxy.mun.ca/10.1007/s002200050822.

[73] S. SMALE, *On the Morse index theorem*, J. Math. Mech., 14 (1965), pp. 1049–1055, https://doi.org/10.1111/j.1467-9876.1965.tb00656.x.

[74] N. G. VAKHITOV AND A. A. KOLOKOLOV, *Stationary solutions of the wave equation in a medium with nonlinearity saturation*, Radiophys. Quantum Electron., 16 (1973), pp. 783–789.