Mean-field approach to finite-size fluctuations in the Kuramoto-Sakaguchi model

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We develop an ab initio approach to describe the statistical behavior of finite-size fluctuations in the Kuramoto-Sakaguchi model. We obtain explicit expressions for the covariance function of fluctuations of the complex order parameter and determine the variance of its magnitude entirely in terms of the equation parameters. Our results rely on an explicit complex-valued formula for solutions of the Adler equation. We present analytical results for both the sub- and the supercritical case. Moreover, our framework does not require any prior knowledge about the structure of the partially synchronized state. We corroborate our results with numerical simulations of the full Kuramoto-Sakaguchi model. The proposed methodology is sufficiently general such that it can be applied to other interacting particle systems.

Introduction. Synchronization is a generic phenomenon inherent in interacting oscillators that determines their ability to exhibit a wide range of complex dynamical behavior [1–7]. It has been observed in a plethora of natural and engineered systems, including pace-maker cells of circadian rhythms [8], networks of neurons [9], chemical oscillators [10, 11] and power grids [12]. The main features of synchronization are described by the paradigmatic Kuramoto-Sakaguchi (KS) model [13]

$$\dot{\psi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\psi_j - \psi_i - \lambda) \tag{1}$$

for the collective behavior of a system of N all-to-all coupled phase oscillators. In this model, ω_i are natural frequencies drawn randomly and independently from a given distribution $g(\omega)$ and K and λ are real parameters quantifying the coupling strength and the phase lag, respectively. It is often convenient to rewrite (1) in the corotating frame $\theta_i = \psi_i - \Omega t$ with a global rotation frequency Ω as

$$\dot{\theta}_i = \omega_i - \Omega + K \operatorname{Im}\left(Z(t)e^{-i\theta_i}e^{-i\lambda}\right), \tag{2}$$

which reveals that each oscillator interacts with the other oscillators only via the mean field. Importantly, the degree of synchronization is quantified by the mean field, also coined the complex order parameter,

$$Z(t) = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}. \tag{3}$$

A fully synchronized state with $\theta_1 = \theta_2 = \cdots = \theta_N$ is characterized by $r(t) = |Z(t)| = 1$, whereas a completely disordered (incoherent) state is characterized by $r(t) \sim \mathcal{O}(1/\sqrt{N})$. It is well known that for sufficiently large values of the coupling strength K the system becomes fully synchronized, whereas for sufficiently small values of K the oscillators behave independently and form an incoherent state. For intermediate values of the coupling strength, partially synchronized states are observed in which $r(t)$ fluctuates around a constant value between 0 and 1. On a microscopic level the oscillator population splits into two groups: coherent oscillators $\mathcal{C}$, which are phase locked, and rogue oscillators $\mathcal{R}$, which drift with respect to the mean field.

In the thermodynamic limit of infinitely many oscillators each oscillator interacts with the other oscillators via a constant collective mean. This limit is well understood and described by mean-field theory [13–16]. In particular, the rotation frequency of the coherent oscillators Ω_∞ and the constant order parameter r_∞ are determined by a self-consistency relation

$$\frac{1}{K}e^{i\lambda} = i\int_{-\infty}^{\infty} g(\Omega_\infty + K r_\infty s) h(s) ds, \tag{4}$$

where

$$h(s) = \begin{cases} (1 - \sqrt{1 - s^{-2}})s & \text{for} \quad |s| > 1 \\ s - i\sqrt{1 - s^2} & \text{for} \quad |s| \le 1. \end{cases} \tag{5}$$

However, recently finite-size effects have gained much attention. Numerical experiments have lead to a better understanding of finite-size effects and provide clear evidence for their significance [17–24]. In particular, several finite-size phenomena were observed such as a stochastic drift of oscillators which disappears in the thermodynamic limit [25–28] and emergent random chimera switching in a deterministic two-population KS model [29]. Capturing finite-size effects analytically is a notoriously hard problem. Relaxation rates and multi-oscillator were obtained using kinetic theory [30–32] and finite-size scaling near criticality has been successfully

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quantified [21, 33–35]. A general scheme for analyzing fluctuations in systems of coupled phase oscillators was proposed in [36–38]. In particular, accurate approximations were obtained in the subcritical case of the KS model (1) with $\lambda = 0$ and a Lorentzian frequency distribution [38]. However, this approach relies on several approximations which do not carry over to the case of more general frequency distributions such as Gaussians as we will show (see End Matter). In this Letter we will capture finite-size effects in a first principle way and provide explicit expressions for the covariance function and the variance of the order parameter. Our focus will be the hitherto unexplored parameter range away from criticality.

Recently, it was numerically recognized that fluctuations

$$\zeta(t) = \sqrt{N}(Z(t) - \langle Z(t)\rangle) \tag{6}$$

around the thermodynamic mean $\langle Z(t)\rangle$, where $\langle\cdot\rangle$ denotes time averaging, approximately obey a Gaussian process [23, 29]. Fluctuations in $Z(t)$, with a variance that decays as $1/N$ and vanishes in the thermodynamic limit, lead to fluctuations in other macroscopic variables, including the order parameter with fluctuations

$$\delta(t) = \sqrt{N}(|Z(t)| - \langle|Z(t)|\rangle) = \sqrt{N}(r(t) - \langle r(t)\rangle). \tag{7}$$

Gaussian processes are entirely determined by their mean, accessible via classical mean-field theory, and their covariance function

$$R(\tau) = \langle\zeta(t)\overline{\zeta(t+\tau)}\rangle, \tag{8}$$

where the bar denotes the complex-conjugate. Once determined, this suggests to approximate the deterministic dynamics of the synchronized oscillators θ_i, which are deterministically driven by $Z(t)$, by an effective stochastic equation for θ_i, which is driven by a Gaussian process [39]. Previous work [23, 29] has shown the efficacy of such an approach, however, these works relied on approximating the general Gaussian process by an Ornstein-Uhlenbeck process the parameters of which were estimated only numerically and needed to be recalculated for each set of equation parameters. Moreover, a precise a priori knowledge of which oscillators are synchronized and which are not was required.

In this Letter, we find explicit analytical expressions for the covariance function (8) and the variance of finite-size fluctuations of the order parameter $V = \langle\delta^2(t)\rangle$, given entirely in terms of equation parameters K, λ and $g(\omega)$. We present expressions for both the partially synchronized and the completely incoherent state. Remarkably, our expressions do not require any a priori knowledge about which oscillators partake in the synchronized cluster and which do not.

Our main results can be formulated as follows. For partially synchronized states we predict

$$R_{\mathrm{ps}}(\tau) = Kr_\infty \int\limits_{-\infty}^{\infty} \frac{(\overline{h^2(s)}-1)(1-|h^2(s)|)g(\Omega_\infty + Kr_\infty s)}{\overline{h^2(s)} - \exp(iKr_\infty s\sqrt{1-s^{-2}}\,\tau)}ds \tag{9}$$

and

$$V_{\mathrm{ps}} = \frac{1}{2}R_{\mathrm{ps}}(0) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \quad \text{for} \quad N \gg 1. \tag{10}$$

For completely incoherent states we find

$$R_{\mathrm{incoh}}(\tau) = \int_{-\infty}^{\infty} g(\omega)e^{-i\omega\tau}d\omega \tag{11}$$

and

$$\sqrt{N}\langle r(t)\rangle = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad V_{\mathrm{incoh}} = 1 - \frac{\pi}{4} \quad \text{for} \quad N \gg 1. \tag{12}$$

Our analytical results and their efficacy in capturing finite-size fluctuations are illustrated in Figs. 1 and 2, where we compare our predictions with simulations of the full KS model (1) for a network of $1,000$ oscillators. Fig. 1 shows the covariance function $R(\tau)$ for a partially synchronized state described by (9) and for a completely incoherent state described by (11). Fig. 2(a) shows the averaged order parameter $\langle r(t)\rangle$ as a function of the coupling strength K. The order parameter of the full KS model (1) is very well reproduced by classical mean-field theory (4) for values of the coupling strength $K \geq K_{\mathrm{crit}}$. Moreover, our extension of the mean-field results to the completely incoherent state given by (12) captures the value of the order parameter for the subcritical range $K < K_{\mathrm{crit}}$. Fig. 2(b) shows how our analytical results for the variance of the order parameter (10), (12) reproduce the true variance well for coupling strength away from the bifurcation at K_{crit}. We have checked that the results for the full KS model (1) are indistinguishable by eye when a larger network with $10,000$ oscillators is simulated. Additional numerical results for various values of the coupling strength K are provided in End Matter.

Covariance function for a partially synchronized state. Our approach is based on the following prerequisites. (i) Although seeking to quantify fluctuations of $Z(t)$, we replace in Eq. (2) the mean-field $Z(t)$ by its constant thermodynamic value Z_∞, and we move into the corotating frame

$$\theta_i(t) = \psi_i(t) - \Omega_\infty t + \phi \quad \text{with some} \quad \phi \in \mathbb{R}. \tag{13}$$

Moreover, we assume $\langle Z(t)\rangle = Z_\infty$. (ii) We use the well-known fact (see End Matter for details) that the probability density of the solution to Eq. (2) with $Z(t) = Z_\infty$ is given by the Ott-Antonson ansatz [16]

$$f_\phi(\theta,\omega) = \frac{g(\omega)}{2\pi}\left\{1 + \sum_{n=1}^{\infty}\left[\overline{z}_\phi^n(\omega)e^{in\theta} + z_\phi^n(\omega)e^{-in\theta}\right]\right\}, \tag{14}$$

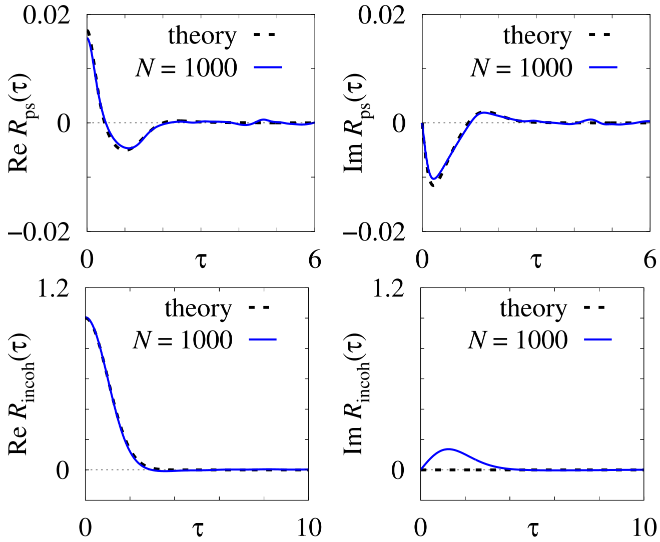

FIG. 1. Covariance $R(\tau)$ of fluctuations $\zeta(t)$ of the complex order parameter for the KS model (1). (Top row) Partially synchronized state at $K = 6$ and $\lambda = \pi/4$, with theoretical prediction (9). (Bottom row) Completely incoherent state at $K = 0.5$ and $\lambda = \pi/4$, with theoretical prediction (11).

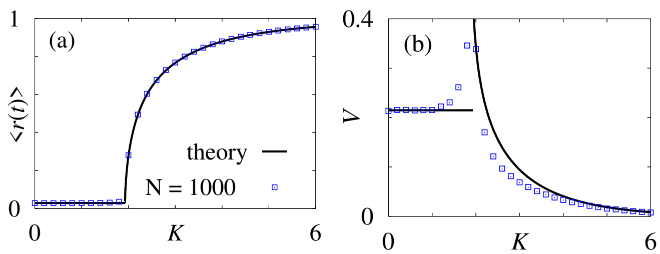

FIG. 2. (a) Averaged order parameter $\langle r(t)\rangle$, with theoretical predictions from classical mean-field theory (4) for $K > K_{\rm crit}$ and from our approach (12) for $K < K_{\rm crit}$, and (b) its variance V, with theoretical predictions (10), (12), versus coupling strength K in the KS model (1) with $\lambda = \pi/4$.

where

$$z_\phi(\omega) = e^{i\phi} h\left(\frac{\omega - \Omega_\infty}{K r_\infty}\right). \tag{15}$$

Then, for every non-negative integer n we have

$$\int_0^{2\pi} \frac{f_\phi(\theta,\omega)}{g(\omega)} e^{in\theta} d\theta = z_\phi^n(\omega), \tag{16}$$

which, upon employing (3) and (4), implies

$$Z_\infty = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} f_\phi(\theta,\omega) e^{i\theta} d\theta = -r_\infty i e^{i\lambda} e^{i\phi}.$$

Moreover, due to the ergodicity property, for any 2π-periodic function $F(\theta)$ we have

$$\langle F(\theta_j(t))\rangle = \int_0^{2\pi} \frac{f_\phi(\theta,\omega_j)}{g(\omega_j)} F(\theta) d\theta. \tag{17}$$

(iii) Finally, we establish an explicit solution of the thermodynamic limit of (2) with constant driver $Z(t) = Z_\infty$, the so called Adler equation. Introducing $\Delta_j = \omega_j - \Omega_\infty$ and $H_\infty = K Z_\infty e^{-i\lambda}$, solutions of the Adler equation $\theta_j(t)$ with initial conditions $e^{i\theta_j(0)}$ are given for $|\Delta_j| \neq |H_\infty|$ by the explicit formula

$$e^{i\theta_j(t)} = W(t, H_\infty, \Delta_j, e^{i\theta_j(0)}), \tag{18}$$

with

$$W(t, H, \Delta, \sigma) = H a_1(t) + \frac{a_2(t)\sigma}{1 + \overline{H} a_1(t)\sigma}, \tag{19}$$

where

$$a_1(t) = i(e^{i\xi t} - 1)/\chi_1(t) \quad \text{and} \quad a_2(t) = 4\xi^2 e^{i\xi t}/\chi_1^2(t)$$

with $\chi_1(t) = (\Delta - \xi)e^{i\xi t} - (\Delta + \xi)$ and

$$\xi = \begin{cases} -i\sqrt{|H|^2 - \Delta^2} & \text{for} \quad |\Delta| \le |H|, \\ \sqrt{\Delta^2 - |H|^2} & \text{for} \quad |\Delta| > |H|. \end{cases}$$

Note that for $|\Delta_j| < |H_\infty|$, the jth oscillator is coherent and the dynamics of $\theta_j(t)$ converges to a fixed point as $t \to \infty$, while for $|\Delta_j| > |H_\infty|$, the oscillator is incoherent with underlying periodic dynamics. Details on the derivation of (18), (19) are given in End Matter.

The covariance function (8) can be expressed in terms of the fluctuations (6) as

$$\begin{aligned} R_{\rm ps}(\tau) &= N\langle Z(t)\overline{Z(t+\tau)}\rangle - N|Z_\infty|^2 \\ &= \frac{1}{N}\sum_{j,k=1}^{N} \langle e^{i\theta_j(t)} e^{-i\theta_k(t+\tau)}\rangle - N|Z_\infty|^2. \end{aligned} \tag{20}$$

It is easy to see that if $j \in \mathcal{C}$ or $k \in \mathcal{C}$, the summands can be written as

$$\langle e^{i\theta_j(t)} e^{-i\theta_k(t+\tau)}\rangle = \langle e^{i\theta_j(t)}\rangle\langle e^{-i\theta_k(t)}\rangle. \tag{21}$$

Moreover, assuming that the variables $e^{i\theta_j(t)}$ and $e^{i\theta_k(t)}$ with $j,k \in \mathcal{R}$ and $j \neq k$ are independent, and therefore identity (21) holds, the sum in (20) can be written as

$$\begin{aligned} \sum_{j,k=1}^{N} \langle e^{i\theta_j(t)} e^{-i\theta_k(t+\tau)}\rangle &= \sum_{j,k=1}^{N} \langle e^{i\theta_j(t)}\rangle\langle e^{-i\theta_k(t)}\rangle \\ &+ \sum_{j\in\mathcal{R}} \langle e^{i\theta_j(t)} e^{-i\theta_j(t+\tau)}\rangle - \sum_{j\in\mathcal{R}} \left|\langle e^{i\theta_j(t)}\rangle\right|^2. \end{aligned}$$

Substitution in (20) yields, upon using (6),

$$R_{\rm ps}(\tau) = \sum_{j\in\mathcal{R}} \langle e^{i\theta_j(t)} e^{-i\theta_j(t+\tau)}\rangle - \sum_{j\in\mathcal{R}} \left|\langle e^{i\theta_j(t)}\rangle\right|^2. \tag{22}$$

Now, using (17) to replace temporal averages by averages over the phases and the explicit formula (18) for $e^{i\theta_j(t+\tau)}$, we obtain

$$\begin{aligned}\langle e^{i\theta_j(t)}\rangle &= z_\phi(\omega_j),\\ \left\langle e^{i\theta_j(t)}e^{-i\theta_j(t+\tau)}\right\rangle &= \int_0^{2\pi} e^{i\theta}\overline{W}(\tau,H_\infty,\Delta_j,e^{i\theta})\times\\ \times\frac{f_\phi(\theta,\omega_j)}{g(\omega_j)}d\theta &= \frac{\overline{Ha_1(\tau)}}{\overline{z_\phi(\omega_j)}}(|z_\phi(\omega_j)|^2-1)+1.\end{aligned}$$

(Note that for the second correlation, we used identity (16) after expanding the solution (18)–(19) in a geometric series. Details of the calculations can be found in End Matter.) Altogether, this yields

$$R_{\text{ps}}(\tau)=\frac{1}{N}\sum_{j=1}^{N}\left[\frac{\overline{Ha_1(\tau)}}{\overline{z_\phi(\omega_j)}}-1\right]\left(|z_\phi(\omega_j)|^2-1\right).$$

Finally, in the thermodynamic limit, averaging over indices j can be replaced by averaging over the distribution $g(\omega)$, so we can write

$$R_{\text{ps}}(\tau)=\int\limits_{-\infty}^{\infty}\left[\frac{\overline{Ha_1(\tau)}}{\overline{z_\phi(\omega)}}-1\right]\left(|z_\phi(\omega)|^2-1\right)g(\omega)d\omega. \tag{23}$$

After some algebraic manipulations (see End Matter), the covariance function (23) simplifies to our expression (9). Note that if $|\omega-\Omega|\le|H_\infty|=Kr_\infty$, $|z_\phi|=1$. Therefore such values ω do not contribute to the integral $R_{\text{ps}}(\tau)$. This has the desirable consequence that in order to evaluate the covariance function, no prior knowledge is required about which oscillators are and which are not synchronized.

Covariance function for a completely incoherent state. Our analysis of the completely incoherent state is based on the following prerequisites. (i) In this state, the phases ψ_j tend to be uniformly distributed on the interval $[0,2\pi]$, and their probability density in the thermodynamic limit is given by

$$f_{\text{incoh}}(\psi,\omega,t)=\frac{g(\omega)}{2\pi}. \tag{24}$$

(ii) The global order parameter shows no collective oscillations, and $Z(t)=0$ as $N\to\infty$. This implies that the dynamics of each oscillator is approximately described by $\dot\psi_i=\omega_i$, and therefore $\psi_i(t)=\omega_i t+\phi$ with $\phi=\psi_i(0)$.

The covariance function for fluctuations $\zeta_{\text{incoh}}(t)$ of this incoherent state is then given by

$$\begin{aligned}R_{\text{incoh}}(\tau) &= \langle\zeta_{\text{incoh}}(t)\overline{\zeta_{\text{incoh}}(t+\tau)}\rangle\\ &= \frac{1}{N}\sum_{j,k=1}^{N}\left\langle e^{i\psi_j(t)}e^{-i\psi_k(t+\tau)}\right\rangle-N|\langle Z(t)\rangle|^2.\end{aligned}$$

For randomly chosen initial conditions $\psi_i(0)$ the variables $e^{i\psi_j(t)}$ and $e^{i\psi_k(t)}$ with $j\neq k$ are independent, which implies again an identity of the form (21), and we obtain

$$R_{\text{incoh}}(\tau)=\frac{1}{N}\sum_{j=1}^{N}\left\langle e^{i\psi_j(t)}e^{-i\psi_j(t+\tau)}\right\rangle-\frac{1}{N}\sum_{j=1}^{N}\left|\langle e^{i\psi_j(t)}\rangle\right|^2.$$

Moreover, using the ergodicity property and the stationary density for incoherent oscillators (24), we find

$$\langle e^{i\psi_j(t)}\rangle=\int_0^{2\pi}e^{i\psi}\frac{f_{\text{incoh}}(\psi,\omega,t)}{g(\omega)}d\psi=0 \tag{25}$$

and

$$\langle e^{i\psi_j(t)}e^{-i\psi_j(t+\tau)}\rangle=e^{-i\omega_j\tau}.$$

Finally, in the large-N limit we can replace averaging over indices j by averaging over the distribution $g(\omega)$ and arrive at our expression for the covariance function R_{incoh} (11). Note that due to the normalization condition for $g(\omega)$, we have $\langle|\zeta_{\text{incoh}}(t)|^2\rangle=R_{\text{incoh}}(0)=1$.

Variance of the order parameter for a partially synchronized state. Rewriting (6) as

$$Z(t)=\langle Z(t)\rangle+\frac{1}{\sqrt{N}}\zeta(t)=Z_\infty+\frac{1}{\sqrt{N}}\zeta(t)$$

implies

$$|Z(t)|^2=|Z_\infty|^2+\frac{2}{\sqrt{N}}\text{Re}\left(\overline{Z_\infty}\zeta(t)\right)+\frac{1}{N}|\zeta(t)|^2, \tag{26}$$

which upon expansion for $N\gg1$ becomes

$$|Z(t)|=|Z_\infty|+\frac{1}{\sqrt{N}|Z_\infty|}\text{Re}\left(\overline{Z_\infty}\zeta(t)\right)+\mathcal{O}\left(\frac{1}{N}\right).$$

This allows to express the fluctuations of the order parameter (7) as

$$\delta(t)=\frac{1}{|Z_\infty|}\text{Re}\left(\overline{Z_\infty}\zeta(t)\right)+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$$

and the variance of the order parameter becomes

$$V_{\text{ps}}=\frac{1}{2}\text{Re}\left(\frac{\overline{Z_\infty^2}}{|Z_\infty|^2}\tilde R_{\text{ps}}(0)\right)+\frac{1}{2}R_{\text{ps}}(0)+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$$

where $\tilde R_{\text{ps}}(t)$ denotes the pseudo-covariance $\tilde R_{\text{ps}}(t)=\langle\zeta(t)\zeta(t+\tau)\rangle$. Employing calculations similar to those for $R_{\text{ps}}(t)$ we can show that $\tilde R_{\text{ps}}(\tau)=0$ for all $\tau\ge0$. The vanishing of the pseudo-covariance can be intuitively understood by evoking a symmetry between the real and imaginary parts of the fluctuations ζ and their independence. Using $\tilde R_{\text{ps}}(\tau)=0$ for all $\tau\ge0$, leads to our result (10).

Variance of the order parameter for a completely incoherent state. Consider fluctuations $\zeta_{\text{incoh}}(t)$ of the complex order parameter in the incoherent state. In the large-N limit, for a completely incoherent state we have $\langle Z(t)\rangle=0$, and (26) reduces to

$$|Z(t)|=\frac{1}{\sqrt{N}}\left|\zeta_{\text{incoh}}(t)\right|.$$

Using the definition of fluctuations (6), we write

$$\zeta_{\rm incoh}(t) = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\cos\psi_j(t) + \frac{i}{\sqrt{N}}\sum_{j=1}^{N}\sin\psi_j(t).$$

In the completely incoherent state all phases $\psi_j(t)$ are uniformly distributed in $[0, 2\pi]$. This implies that in the large-N limit fluctuations satisfy a central limit theorem, and $\zeta_{\rm incoh}(t)$ is a two-dimensional random vector distributed according to a bivariate normal distribution. The mean and variances are readily determined. For each $t \geq 0$ the function $\cos\psi_j(t)$ is a random variable with zero expected value $\langle\cos\psi_j\rangle_{\rm ens} = 0$ and variance

$$\left\langle\cos^2\psi_j\right\rangle_{\rm ens} = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\psi\, d\psi = \frac{1}{2},$$

where $\langle\cdot\rangle_{\rm ens}$ denotes the ensemble average. Similarly, $\sin\psi_j(t) \sim \mathcal{N}(0, 1/2)$. Moreover, it is easy to verify that the random variables $\cos\psi_j(t)$ and $\sin\psi_j(t)$ are uncorrelated, since

$$\langle\cos\psi_j\sin\psi_j\rangle_{\rm ens} = \frac{1}{2\pi}\int_0^{2\pi}\cos\psi\sin\psi\, d\psi = 0.$$

Hence Re $\zeta_{\rm incoh}(t) \sim \mathcal{N}(0, 1/2)$ and Im $\zeta_{\rm incoh}(t) \sim \mathcal{N}(0, 1/2)$, and $|\zeta_{\rm incoh}(t)|$ has a Rayleigh distribution with expected value

$$\langle|\zeta_{\rm incoh}(t)|\rangle = \sqrt{\pi}/2$$

and variance

$$V_{\rm incoh} = \langle|\zeta_{\rm incoh}(t)|^2\rangle - \langle|\zeta_{\rm incoh}(t)|\rangle^2 = 1 - \pi/4.$$

Note that the value $\langle|\zeta_{\rm incoh}(t)|^2\rangle = 1$, which follows from the last two identities, agrees with the value $\langle|\zeta_{\rm incoh}(t)|^2\rangle = R_{\rm incoh}(0)$ obtained above.

Discussion and outlook. In this Letter, we have proposed a fully analytical approach to capture the statistical behavior of finite-size fluctuations in the KS model (1). In particular, our approach provides

(i) expressions for the covariance function of fluctuations of the complex order parameter and the variance of the order parameter entirely in terms of the KS model parameters,

(ii) expressions for both sub- and super-critical coupling strengths,

(iii) expressions that do not require prior knowledge about which oscillators are synchronized and which are not.

The results obtained were verified with simulations for the full KS model and were shown to perform well away from criticality. On the other hand, our approximations were less accurate when approaching the critical coupling strength $K_{\rm crit} \approx 1.92$ demarcating the transition from a desynchronized to a partially synchronized state. It seems that near criticality, long-range correlations cannot be neglected and our independence assumptions cease to be valid. We envisage extensions of our framework to the near critical case using a perturbative approach around the expressions derived in this work.

Our approach is sufficiently general to allow for further applications to other interacting particle systems such as θ-neuron models and more complex network topologies going beyond the all-to-all coupling considered in this Letter. The mathematical tools developed here can potentially be adapted to study fluctuations in coupled oscillator systems in the presence of noise [28, 40–44] and finite-size corrections of Lyapunov exponents [45].

In summary, our results fully specify the statistical behavior of finite-size fluctuations in the KS model which have been numerically shown to be Gaussian processes. This opens up the way to perform systematic stochastic model reductions for such interacting particle systems from first principles and to construct reduced stochastic equations for designated collective variables [27, 46].

END MATTER

Thermodynamic limit for the KS model (1). Most of the dynamical states found in the KS model (1) with finite N, have complex chaotic behavior. However, in the thermodynamic limit $N \to \infty$, they admit a simple analytic representation. If $N \gg 1$, the state of the phase oscillators $\{\psi_i(t)\}$ can be described by a probability density function $f(\psi, \omega, t)$, which obeys the continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial\psi}(fv) = 0, \tag{27}$$

where

$$v(\psi, \omega, t) = \omega + \frac{K}{2i}\left[e^{-i\lambda}\mathcal{Z}(t)e^{-i\psi} - e^{i\lambda}\overline{\mathcal{Z}}(t)e^{i\psi}\right]$$

is the continuum version of the velocity field in Eq. (1), and

$$\mathcal{Z}(t) = \int_{-\infty}^{\infty}d\omega\int_0^{2\pi}f(\psi, \omega, t)e^{i\psi}d\psi \tag{28}$$

is the continuum version of the complex order parameter (see formula (3) with $\Omega = 0$).

It is well-known [16, 47] that the long-term dynamics of the continuity equation (27) asymptotically approaches the Ott-Antonsen manifold for $t \to \infty$, consisting of distributions of the form

$$f(\psi, \omega, t) = \frac{g(\omega)}{2\pi}\left\{1 + \sum_{n=1}^{\infty}\left[\overline{z}^n(\omega, t)e^{in\psi} + z^n(\omega, t)e^{-in\psi}\right]\right\},$$

where $z(\omega, t)$ satisfies the inequality $|z| \leq 1$ and solves the integro-differential equation

$$\frac{\partial z}{\partial t} = i\omega z(\omega, t) + \frac{K}{2}e^{-i\lambda}\mathcal{G}z - \frac{K}{2}e^{i\lambda}z^2(\omega, t)\mathcal{G}\overline{z} \tag{29}$$

with the integral operator

$$(\mathcal{G}z)(t) = \int_{-\infty}^{\infty} g(\omega) z(\omega, t) d\omega.$$

Remarkably, all statistically stationary states of system (1) lie on the Ott-Antonsen manifold [14, 15]. In particular, the zero solution $z(\omega, t) = 0$ of Eq. (29) corresponds to the completely incoherent state in (1), while all stationary partially synchronized states in (1) are represented by rotating waves

$$z(\omega, t) = h\left(\frac{\omega - \Omega_\infty}{K r_\infty}\right) e^{i\Omega_\infty t} \tag{30}$$

where $h(s)$ is defined by (5) and $(r_\infty, \Omega_\infty) \in (0,1) \times \mathbb{R}$ is a pair of numbers satisfying the self-consistency equation (4).

The physical meaning of the parameters r_∞ and Ω_∞ can be understood by inserting the probability density $f(\psi, \omega, t)$ corresponding to (30) into Eq. (28). This yields

$$\mathcal{Z}(t) = -r_\infty i e^{i\lambda} e^{i\Omega_\infty t}, \tag{31}$$

and therefore

$$|\mathcal{Z}(t)| = r_\infty \tag{32}$$

is the magnitude of the order parameter, while Ω_∞ is the angular speed of its phase.

If in the KS model (1) we move into the corotating frame of reference (13), the probability density function in the new frame will take on the time-independent form (14), which is used for the calculations in the main text.

Explicit solutions of the Adler equation Let us consider the Adler equation written in complex form,

$$\dot{\theta} = \Delta + \mathrm{Im}\left(H e^{-i\theta}\right) = \Delta + \frac{H e^{-i\theta} - \overline{H} e^{i\theta}}{2i} \tag{33}$$

with arbitrary $\Delta \in \mathbb{R}$ and $H \in \mathbb{C}$.

Proposition 1 *If $|\Delta| \neq |H|$, the solution of the Adler equation (33) is of the form*

$$e^{i\theta(t)} = W(t, H, \Delta, e^{i\theta(0)}) := \frac{iH(e^{i\xi t} - 1) - \chi_2(t) e^{i\theta(0)}}{i\overline{H}(e^{i\xi t} - 1) e^{i\theta(0)} + \chi_1(t)},$$

where

$$\xi = \begin{cases} -i\sqrt{|H|^2 - \Delta^2} & \text{for} \quad |\Delta| \leq |H|, \\ \sqrt{\Delta^2 - |H|^2} & \text{for} \quad |\Delta| > |H|, \end{cases} \tag{34}$$

and

$$\chi_1(t) = (\Delta - \xi) e^{i\xi t} - (\Delta + \xi), \quad \chi_2(t) = (\Delta + \xi) e^{i\xi t} - (\Delta - \xi).$$

In addition, in the degenerate case $|\Delta| = |H|$, the solution of the Adler equation (33) is given by

$$e^{i\theta(t)} = W_0(t, H, \Delta, e^{i\theta(0)}) := z_* + \frac{e^{i\theta(0)} - z_*}{1 + (e^{i\theta(0)} - z_*)\overline{H} t/2},$$

where $z_* = i\Delta/H$. (For brevity of notations, we do not specify that ξ, $\chi_1(t)$, $\chi_2(t)$ and z_* depend on Δ and H.)

Proof: Multiplying Eq. (33) by $ie^{i\theta}$, we obtain a complex differential equation for $z = e^{i\theta}$:

$$\dot{z} = \frac{1}{2} H + i\Delta z - \frac{1}{2} \overline{H} z^2. \tag{35}$$

For $|\Delta| \neq |H|$, the right-hand side of this equation can be written as

$$\frac{1}{2} H + i\Delta z - \frac{1}{2} \overline{H} z^2 = -\frac{1}{2} \overline{H} (z - z_+)(z - z_-),$$

where $z_\pm = i(\Delta \pm \xi)/\overline{H}$. Using the method of separation of variables and the initial condition $z(0) = \sigma$, we obtain

$$z(t) = \frac{z_+(\sigma - z_-) e^{i\xi t} - z_-(\sigma - z_+)}{(\sigma - z_-) e^{i\xi t} - (\sigma - z_+)},$$

or equivalently $e^{i\theta(t)} = W(t, H, \Delta, e^{i\theta(0)})$.

In the degenerate case $|\Delta| = |H|$, we have $z_+ = z_- = z_*$. Integrating Eq. (35), we obtain $e^{i\theta(t)} = W_0(t, H, \Delta, e^{i\theta(0)})$. ■

Remark 1 *In the case $|\Delta| \neq |H|$, we can write*

$$W(t, H, \Delta, \sigma) = H a_1(t) + \frac{a_2(t)\sigma}{1 + \overline{H} a_1(t) \sigma},$$

with

$$a_1(t) = \frac{i(e^{i\xi t} - 1)}{\chi_1(t)} \quad \text{and} \quad a_2(t) = \frac{4\xi^2 e^{i\xi t}}{\chi_1^2(t)}.$$

Moreover, in this case, we have $|H a_1(t)| < 1$, and hence

$$W(t, H, \Delta, \sigma) = H a_1(t) + a_2(t) \sum_{n=0}^{\infty} (-1)^n \overline{H}^n a_1^n(t) \sigma^{n+1}$$

is an absolutely convergent series for $|\sigma| \leq 1$.

Remark 2 *If $(r_\infty, \Omega_\infty) \in (0,1) \times \mathbb{R}$ satisfy the self-consistency equation (4), it can be easily verified that $z_\varphi(\omega)$ defined by (15) is a fixed point of the Adler equation (33) with $\Delta = \omega - \Omega_\infty$ and $H = -iKr_\infty e^{i\phi}$. Therefore,*

$$W\left(t, -iKr_\infty e^{i\phi}, \omega - \Omega_\infty, z_\phi(\omega)\right) = z_\phi(\omega)$$

for all $t, \omega, \phi \in \mathbb{R}$.

Remark 3 *Under the assumptions of Remark 2, using*

Remark 1 and formula (16)*, we can show*

$$\int_0^{2\pi} e^{i\theta}\overline{W}(t,H,\Delta,e^{i\theta})\frac{f_\phi(\theta,\omega)}{g(\omega)}d\theta = \int_0^{2\pi}\left[\overline{Ha_1(t)}e^{i\theta}\right.$$
$$\left.+\overline{a_2(t)}\sum_{n=0}^{\infty}(-1)^n H^n\overline{a_1^n(t)}e^{-in\theta}\right]\frac{f_\phi(\theta,\omega)}{g(\omega)}d\theta$$
$$=\overline{Ha_1(t)}z_\phi(\omega)+\overline{a_2(t)}\sum_{n=0}^{\infty}(-1)^nH^n\overline{a_1^n(t)}\overline{z}_\phi^n(\omega)$$
$$=\overline{Ha_1(t)}\left(z_\phi(\omega)-\frac{1}{\overline{z_\phi(\omega)}}\right)+\frac{1}{\overline{z_\phi(\omega)}}\overline{W(t,H,\Delta,z_\phi(\omega))}$$
$$=\frac{\overline{Ha_1(t)}}{\overline{z_\phi(\omega)}}(|z_\phi(\omega)|^2-1)+1.$$

Derivation of Eq. (9) **from Eq.** (23)**.** Using (5) it is easy to verify that

$$|H|h\left(\frac{\Delta}{|H|}\right)=\begin{cases}\Delta-\xi\,\mathrm{sgn}(\Delta) & \text{for}\quad |\Delta|>|H|,\\ \Delta+\xi & \text{for}\quad |\Delta|\le|H|,\end{cases}\tag{36}$$

where ξ is given by (34). Note that formula (36) holds for all $\Delta\in\mathbb{R}$ and $H\in\mathbb{C}$.

Now, we consider the integrand of Eq. (23) recalling that $H=KZ_\infty e^{-i\lambda}=-iKr_\infty e^{i\phi}$. Recognizing that in this case

$$\frac{H}{z_\phi(\omega)}=-\frac{iKr_\infty e^{i\phi}}{e^{i\phi}h\left(\dfrac{\omega-\Omega_\infty}{Kr_\infty}\right)}=-\frac{i|H|}{h\left(\dfrac{\Delta}{|H|}\right)}$$

does not depend on ϕ, we find

$$\frac{Ha_1(\tau)}{z_\phi(\omega)}-1=-\frac{i|H|}{h\left(\dfrac{\Delta}{|H|}\right)}\frac{i(e^{i\xi t}-1)}{\chi_1(t)}-1$$
$$=\frac{|H|^2}{|H|h\left(\dfrac{\Delta}{|H|}\right)}\frac{e^{i\xi t}-1}{(\Delta-\xi)e^{i\xi t}-(\Delta+\xi)}-1.\tag{37}$$

For $\Delta>|H|$, upon using (36) and the identity $\Delta^2-\xi^2=|H|^2$, (37) is written as

$$\frac{Ha_1(\tau)}{z_\phi(\omega)}-1=\frac{|H|^2(e^{i\xi t}-1)}{(\Delta-\xi)^2e^{i\xi t}-(\Delta+\xi)(\Delta-\xi)}-1$$
$$=\frac{e^{i\xi t}-1}{h^2(\Delta/|H|)e^{i\xi t}-1}-1=\frac{1-h^2(\Delta/|H|)}{h^2(\Delta/|H|)-e^{-i\xi t}}.$$

Similarly, for $\Delta\le|H|$ we obtain

$$\frac{Ha_1(\tau)}{z_\phi(\omega)}-1=\frac{1-h^2(\Delta/|H|)}{h^2(\Delta/|H|)-e^{i\xi t}},$$

and thus in both cases we have

$$\frac{Ha_1(\tau)}{z_\phi(\omega)}-1=\frac{1-h^2(\Delta/|H|)}{h^2(\Delta/|H|)-e^{-i\Delta\sqrt{1-|H|^2/\Delta^2}t}}.$$

Substituting into Eq. (23), we obtain Eq. (9).

Pseudo-covariance function for a partially synchronized state. Using the definition of the pseudo-covariance function $\tilde{R}_{\mathrm{ps}}(t)=\langle\zeta(t)\zeta(t+\tau)\rangle$ and assuming that the variables $e^{i\theta_j(t)}$ and $e^{i\theta_k(t)}$ are independent for $j\ne k$, by analogy with (22), we obtain

$$\tilde{R}_{\mathrm{ps}}(\tau)=\sum_{j\in\mathcal{R}}\langle e^{i\theta_j(t)}e^{i\theta_j(t+\tau)}\rangle-\sum_{j\in\mathcal{R}}\langle e^{i\theta_j(t)}\rangle^2.$$

Using the ergodicity property (17) and performing calculations as in Remark 3, we find

$$\langle e^{i\theta_j(t)}e^{i\theta_j(t+\tau)}\rangle=\int_0^{2\pi}e^{i\theta}W(\tau,H_\infty,\Delta_j,e^{i\theta})\frac{f_\phi(\theta,\omega_j)}{g(\omega_j)}d\theta$$
$$=z_\phi(\omega_j)W(\tau,H_\infty,\Delta_j,z_\phi(\omega_j))=z_\phi^2(\omega_j).$$

Recalling that $\langle e^{i\theta_j(t)}\rangle=z_\phi(\omega_j)$, we obtain $\tilde{R}_{\mathrm{ps}}(\tau)=0$. Note that this relation is valid regardless of the choice of ϕ in the definition of the corotating frame (13).

Numerical simulation protocol for system (1)**.** Given a distribution $g(\omega)$, we generate a set of N natural frequencies ω_j according to

$$\int_{-\infty}^{\omega_j}g(\omega)d\omega=\frac{j}{N+1},\qquad j=1,\dots,N.$$

This equiprobable sampling avoids finite-size effects such as randomly clustered natural frequencies leading to small synchronized clusters with non-zero mean frequencies and their respective interactions [48, 49]. For each pair of parameters K and λ, we perform simulations using a standard Runge-Kutta solver with constant time step $dt=0.02$. We start from initial conditions chosen randomly from the interval $[0,2\pi]$, discard a transient period of $T_{\mathrm{transient}}=10^5$ time units and use the subsequent interval of length $T_{\max}=10^5$ time units to calculate statistical quantities such as means, variances, and covariances.

Additional numerical results. To demonstrate how the accuracy of our method changes as we approach the critical coupling strength $K_{\mathrm{crit}}\approx 1.92$, we show additional results for partially synchronized states at $K=5,4,3,2$ for $\lambda=\pi/4$, see Figs. 3–6, as well as for a completely incoherent state at $K=1$ and $\lambda=\pi/4$, see Fig. 7.

Daido's theory. In [38], H. Daido proposed an analytical approach to approximate finite-size fluctuations in the KS model (1) with $\lambda=0$. In particular, for the subcritical case he obtained explicit expressions for the covariance function and the variance of the complex order

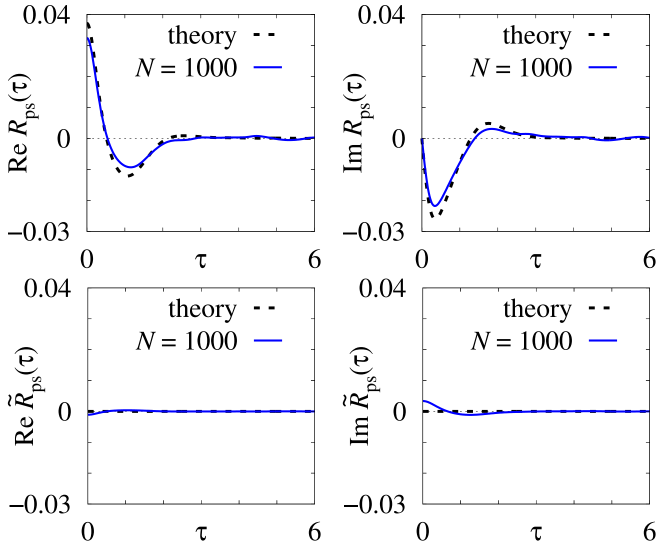

FIG. 3. Covariance $R(\tau)$ and pseudo-covariance $\tilde{R}(\tau)$ of the complex order parameter fluctuation $\zeta(t)$ for a partially synchronized state at $K = 5$ and $\lambda = \pi/4$ in the KS model (1). Numerical simulations (solid curve) vs. theoretical prediction (9) (dashed curve).

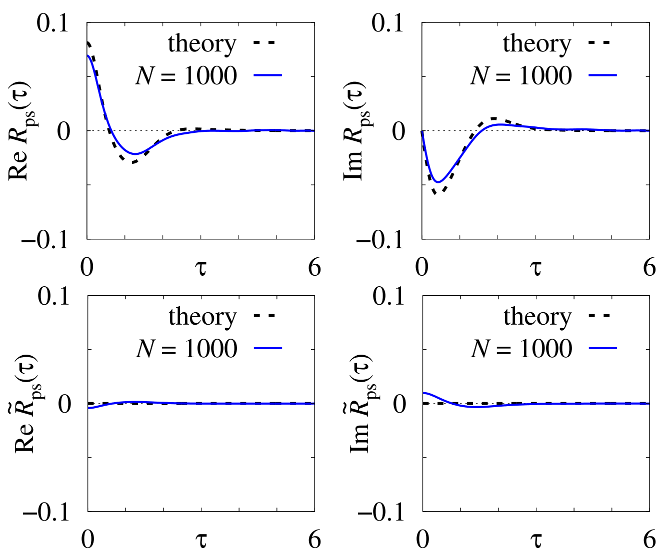

FIG. 4. The same fluctuation characteristics as in Fig. 3 but for a partially synchronized state at $K = 4$ and $\lambda = \pi/4$.

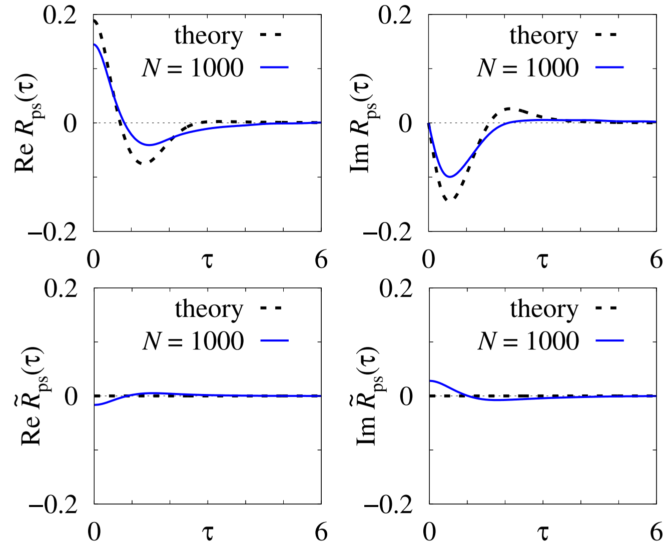

FIG. 5. The same fluctuation characteristics as in Fig. 3 but for a partially synchronized state at $K = 3$ and $\lambda = \pi/4$.

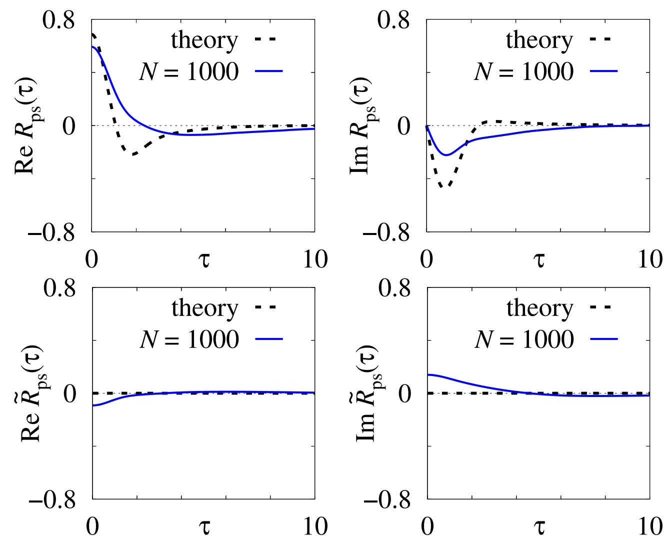

FIG. 6. The same fluctuation characteristics as in Fig. 3 but for a partially synchronized state at $K = 2$ and $\lambda = \pi/4$.

parameter. We follow here his approach and determine expressions for $\lambda \neq 0$ in the subcritical case, and show that unlike for a Lorentz distribution of the natural frequencies this method is not well suited for a Gaussian frequency distribution.

Let us assume that the dynamics of the ith oscillator can be represented as

$$\psi_i(t) = \Theta_i(t) + \frac{1}{\sqrt{N}}\vartheta_i(t), \tag{38}$$

where $\Theta_i, \vartheta_i = \mathcal{O}(1)$ for $N \to \infty$. Upon substitution into Eq. (2) with $\Omega = 0$ in the subcritical case, we obtain

$$\dot{\Theta}_i + \frac{1}{\sqrt{N}}\dot{\vartheta}_i = \omega_i + K\,\mathrm{Im}\left(Z(t)e^{-i(\Theta_i+\vartheta_i/\sqrt{N})}e^{-i\lambda}\right).$$

Next, assuming that $\langle Z(t)\rangle = 0$ and hence $Z(t) = \mathcal{O}(1/\sqrt{N})$, and equating separately the terms of order $\mathcal{O}(1)$ and the terms of order $\mathcal{O}(1/\sqrt{N})$, we obtain

$$\dot{\Theta}_i = \omega_i \tag{39}$$

and

$$\frac{1}{\sqrt{N}}\dot{\vartheta}_i = K\,\mathrm{Im}\left(Z(t)e^{-i\Theta_i}e^{-i\lambda}\right). \tag{40}$$

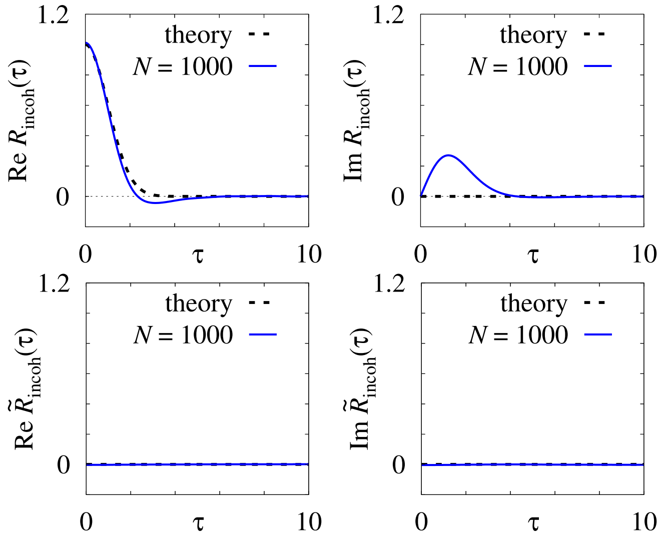

FIG. 7. Covariance $R(\tau)$ and pseudo-covariance $\tilde{R}(\tau)$ of the complex order parameter fluctuation $\zeta(t)$ for a completely incoherent state at $K = 1$ and $\lambda = \pi/4$ in the KS model (1). Numerical simulations (solid curve) vs. theoretical prediction (11) (dashed curve).

Eq. (39) is solved by $\Theta_i(t) = \omega_i t + \Theta_i(0)$. Substituting this into Eq. (40), we obtain

$$\frac{1}{\sqrt{N}}\dot{\vartheta}_i(t) = K\int_0^t \mathrm{Im}\left(Z(t')e^{-i(\omega_i t'+\Theta_i(0))}e^{-i\lambda}\right)dt'$$

with $\vartheta_i(0) = 0$. Finally, substituting the expressions for the solution to Eqs. (39) and (40) into (38) and then substituting $\theta_i(t) = \psi_i(t)$ into the definition of the complex order parameter (3), we obtain a self-consistency equation for $Z(t)$:

$$\begin{aligned} Z(t) &= \frac{1}{N}\sum_{j=1}^{N} e^{i(\omega_i t+\Theta_i(0))}\times \qquad (41)\\ &\times\left(1+iK\int_0^t \mathrm{Im}\left(Z(t')e^{-i(\omega_i t'+\Theta_i(0))}e^{-i\lambda}\right)dt'\right)\\ &= \frac{1}{N}\sum_{j=1}^{N} e^{i(\omega_j t+\Theta_j(0))}\\ &+ \frac{K}{2}e^{-i\lambda}\int_0^t Z(t')\frac{1}{N}\sum_{j=1}^{N} e^{i\omega_j(t-t')}dt'\\ &- \frac{K}{2}e^{i\lambda}\int_0^t \overline{Z}(t')\frac{1}{N}\sum_{j=1}^{N} e^{i(\omega_j(t+t')+2\Theta_j(0))}dt', \qquad (42)\end{aligned}$$

where we employed a Taylor expansion of the exponential function for $|Z(t)| \ll 1$.

In the next step, we will obtain an asymptotic formula for $Z(t)$ as $t \to \infty$. For this, we first notice that the function

$$\frac{1}{N}\sum_{j=1}^{N} e^{i(\omega_j(t+t')+2\Theta_j(0))}$$

is vanishing as $t \to \infty$ and hence can be neglected. Furthermore, for $N \gg 1$ we have

$$\frac{1}{N}\sum_{j=1}^{N} e^{i\omega_j t} \mapsto \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega = \hat{g}(t).$$

Thus, the complex order parameter $Z(t)$ given in (42) is the solution of the following Volterra integral equation

$$\begin{aligned} Z(t) &= \frac{1}{N}\sum_{j=1}^{N} e^{i(\omega_j t+\Theta_j(0))}\\ &+ \frac{K}{2}e^{-i\lambda}\int_0^t \hat{g}(t-t')Z(t')dt'. \qquad (43)\end{aligned}$$

This equation can be solved using the Laplace transformation

$$\mathcal{L}\,:\, f(t) \mapsto F(s) = \int_0^{\infty} e^{-st}f(t)dt.$$

In particular, due to the convolution theorem, Eq. (43) we write

$$(\mathcal{L}Z)(s) = \frac{1}{N}\sum_{j=1}^{N}\frac{e^{i\Theta_j(0)}}{s-i\omega_j} + \frac{K}{2}e^{-i\lambda}(\mathcal{L}\hat{g})(s)(\mathcal{L}Z)(s),$$

and therefore obtain

$$(\mathcal{L}Z)(s) = \frac{1}{N}\sum_{j=1}^{N}\left(1-\frac{K}{2}e^{-i\lambda}(\mathcal{L}\hat{g})(s)\right)^{-1}\frac{e^{i\Theta_j(0)}}{s-i\omega_j}.$$

Now, assuming that

$$Q(s) := 1-\frac{K}{2}e^{-i\lambda}(\mathcal{L}\hat{g})(s) \neq 0 \quad \text{for all} \quad \mathrm{Re}(s) \geq 0$$

and that $1/Q(s)$ is a meromorphic function in the half-plane $\mathrm{Re}(s) < 0$ that satisfies

$$\lim_{p\to\infty}\max_{\pi/2\leq|\phi|\leq\pi}|1/Q(pe^{i\phi})| = 0, \qquad (44)$$

we apply the inverse Laplace transformation and find that the long-term asymptotics of $Z(t)$ is given by

$$Z(t) = \frac{1}{N}\sum_{j=1}^{N}\frac{e^{i(\omega_j t+\Theta_j(0))}}{Q(i\omega_j)}.$$

The covariance function of the complex order parameter $Z(t)$ is now evaluated as

$$\begin{aligned} R_{\rm Daido}(\tau) &= N\langle Z(t)\overline{Z(t+\tau)}\rangle\\ &= \frac{1}{N}\sum_{j,k=1}^{N}\frac{e^{-i(\omega_k\tau+\Theta_k(0)-\Theta_j(0))}}{Q(i\omega_j)\overline{Q(i\omega_k)}}\langle e^{i(\omega_j-\omega_k)t}\rangle\\ &= \frac{1}{N}\sum_{j=1}^{N}\frac{e^{-i\omega_j\tau}}{|Q(i\omega_j)|^2} \approx \int_{-\infty}^{\infty} g(\omega)\frac{e^{-i\omega\tau}}{|Q(i\omega)|^2}d\omega,\end{aligned}$$

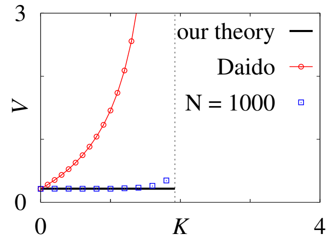

FIG. 8. Variance V of the order parameter $r(t)$ versus coupling strength K in the subcritical case of the KS model (1) with $\lambda = \pi/4$ and a Gaussian distribution. Theoretical predictions from our approach (12) (thick line) and from Daido's theory (45) (circles) are compared with the results of numerical simulations with $N = 1,000$ phase oscillators (squares). The vertical dashed line indicates the critical coupling strength.

where we used $\langle e^{i(\omega_j - \omega_k)t} \rangle = \delta_{jk}$. The variance of the order parameter $r(t) = |Z(t)|$ in the subcritical case is then given by

$$V_{\mathrm{Daido}} = N\langle |Z(t)|^2 \rangle - N\langle |Z(t)| \rangle^2 = R_{\mathrm{Daido}}(0) - \frac{\pi}{4}, \quad (45)$$

where we used our result (12) for $\langle |Z(t)| \rangle$. Although the above formulas have been shown to be in good agreement with the numerical observations obtained from the full KS model (1) with a Lorentzian frequency distribution and $\lambda = 0$, in the case of a Gaussian frequency distribution they fail to reproduce the observed behavior as seen in Fig. 8. In particular, they can be worse than our (K, λ)-independent formulas (11) and (12). Note that condition (44) is not satisfied for a Gaussian frequency distribution.

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