Math. Scientist **33**, 1–7 (2008)
Printed in England

IN VINO VERITAS, IN DOLIO CALAMITAS

FRANÇOIS JONGMANS,* *Université de Liège*

Abstract

We review five classical geometrical models for the volume of a barrel, four of which go back at least to Johannes Kepler in the 17th century. The fifth model, proposed by Charles Camus in 1741, indicated a fruitful new direction, but was superseded by a number of 'empirical' formulae for volume, some still used by wine gaugers today. These are generally inaccurate and/or do not permit a logical extension to partly filled barrels. We propose three new geometrical models motivated by Camus' ideas. All eight volume expressions are shown, using integral calculus, to have a common structure. The physical measurement is most easily made by a rod inserted vertically through the bunghole, followed by a simple calculation.

Keywords: Barrel; geometric model; volume; conic sections; concave function; curvature; weighted mean; velte

2000 Mathematics Subject Classification: Primary 52-03
Secondary 01A50; 01A55

1. Introduction

This paper is concerned with efforts over time to measure the internal volume of wine contained in a full *barrel* or *cask* (*dolium* in Latin; *tonneau*, *barrique*, or *fût* in French), an assembly of several curved wooden staves and two flat ends. The measuring procedure may be called *barrel gauging* after the tool called a *gauge* (*jauge* in French). This is a simple rod of wood or metal which is plunged vertically or diagonally into a barrel lying on its side (see Figure 1 for a profile) through the *bunghole* (*bonde* in French), an opening in the barrel at the highest point on the opposite side.

The measurement thus made (the length $D = 2R$ of CC′ or k of CE in Figure 1) is used in conjunction with the length, $l = 2a$, of the barrel (that is, of A′A), and the diameter, $d = 2r$, of the ends (that is, the length of BE), to give an expression for the volume, V, of a full barrel by way of a generally 'empirical' formula. Since, clearly, $k^2 = a^2 + (R+r)^2 = l^2/4 + (R+r)^2$ (by Pythagoras' theorem) any such formula for volume can be expressed in terms of l, R, and r.

Another parameter of interest to which we shall refer is $\tau = R/r$, self-evidently named the *swelling number*. A swelling number of 1.25, while common in Marseille, is generally considered high for most barrels.

The chief tool for wine gauging of small barrels during the 19th and 20th centuries seems to have been the 'velte', earlier called *kubische Visierruthe* in German countries and *diagonael vergierroede* in Flanders. This typical rod was already known in Flanders around the middle of the 16th century. This is a 'diagonal' rod thrust in the direction CE, and originally graduated so that the gauger could read the volume directly on the velte. The main champion of this tool was the famous astronomer Johannes Kepler (1571–1630), who was astonished in 1613 by the quick gauging of several barrels with a single rod at his second wedding in Linz, Austria.

Received 18 July 2006; revision received 4 January 2007.
* Current address: Trix du Vieux Mayeur 16, B4670 Mortier, Belgium. Email address: eseneta@maths.usyd.edu.au

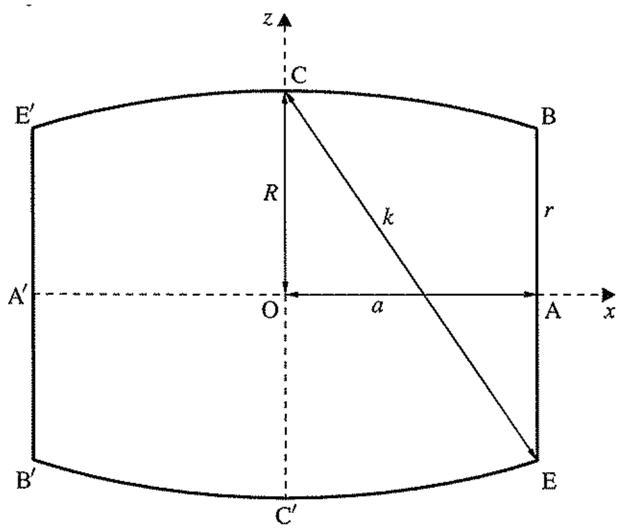

FIGURE 1: A barrel lying on its side.

Nowadays the graduation on the diagonal rod is linear, and an empirical expression for the volume W_1 of the liquid in a full barrel is given by

$$W_1 = qk^3 = q\left[\frac{l^2}{4} + (R+r)^2\right]^{3/2}, \tag{1}$$

with a suitable coefficient q for which the French Office of Customs and Excise has proposed a value $\frac{3}{5} = 0.6$. The measurements are in decimetres for lengths and litres for volume. This formula, while needing only a simple calculation once the *diagonal* measurement k is obtained, does not always give a good approximation to the actual volume, as we shall see later.

A strong impulse for scientifically based gauging came from Kepler. He devoted two short books in 1615 (in Latin) and in 1616 (in German) to the topic (see [4]). He tried to find the volume of several typical barrels, and more generally of bodies obtained by rotation of a contour (upper profile) around an axis. The advent of the integral calculus which swept Europe soon after, facilitated the calculation of such a volume. Various models for the upper profile could thus be treated easily, but the results were disappointing, giving values for volumes which were too small or too large for the barrels they were supposed to represent. We shall consider four such expressions for volume from profiles listed by Kepler.

The fifth model to be considered is due to a somewhat less famous French scientist, Charles Camus (1699–1768), and was presented by him in 1741 [1]. The remaining three models out of a total of eight are the author's own, but using only the simple calculus tools already available 250 years ago. In the discussion to follow, all volumes, both model and 'empirical', are given by an expression of the following form as the product of πl with a 'weighted mean':

$$V = \pi l(\alpha R^2 + \beta Rr + \gamma r^2), \tag{2}$$

with the coefficients, α, β, and γ (positive, null, or negative) summing up to 1. Such expressions for volume are therefore appropriate for the use of a *vertical* linearly marked measuring rod to read the value of $D = 2R$ directly.

For our eight model volumes, we use the notation $V_i, i = 1, 2, \ldots, 8$. A comparison between two such expressions, V_i and V_j, for a given *fundamental triplet* $(l = 2a, D = 2R, d = 2r)$ may conveniently be made by taking the difference $V_i - V_j$, which will be of the general form $\pi l(\lambda R^2 + \mu Rr + \nu r^2)$, $\lambda + \mu + \nu = 0$, which can be factorized as

$$V_i - V_j = \pi l(R - r)(\lambda R - \nu r), \tag{3}$$

from which any difference is easily evaluated.

The numerical values of subscripts of the V_is will correspond, in the sequel, to increasing values of the V_is.

An explanation for the unified structure, expressed by (2) and (3), of model V_is is given in Appendix A.

Camus, by insisting that the upper profile satisfies certain smoothness conditions, was a pioneer, heralding a fruitful theoretical approach after a period of stagnation. However, Camus' ideas were corrupted in the late 1700s by Dez [2], [3]. Then the French Revolution of 1789 initiated an empiricism devoid of elementary mathematics.

This is the reason for the title of the present paper, which can be loosely translated from the Latin as: 'Truth in wine, but disaster in the wine barrel'.

2. Classical geometric models

Referring to Figure 1, a barrel T is conceived as a hollow body obtained by rotation, around the x-axis, of a contour BCE′. The barrel consists of an upper profile BE′ and a lower profile B′E, with two vertical segments on the left and right sides in the vertical plane xOz. The essential feature of the model is the equation $z = f(x)$ for the upper profile BE′, in terms of a function f which is concave, symmetric, and continuous on the closed interval $[-a, a]$, with $f(a) = f(-a) = r$. The number $R = f(0)$ is the radius of the *bulge* (*bouge* in French).

The volume of our model is

$$V = \int_{-a}^{a} S(x)\,\mathrm{d}x = 2\int_{0}^{a} S(x)\,\mathrm{d}x,$$

where $S(x)$ is the area $\pi f^2(x)$ of the circle described by the point (x, z) of the profile when rotated about the x-axis. So,

$$V = 2\pi \int_{0}^{a} f^2(x)\,\mathrm{d}x. \tag{4}$$

The first item from Kepler's list which we consider, namely two truncated cones placed back to back, had already been mentioned in the first century of our era by Heron of Alexandria, with the volume correctly calculated well before Kepler. It corresponds to

$$f(x) = R - \frac{R - r}{a}|x|,$$

with the upper profile being the straight line segments joining E′C and CB. Then (4) gives

$$V_1 = \pi l \frac{R^2 + Rr + r^2}{3}. \tag{5}$$

Clearly this is the minimal volume for any model of the kind we have specified, for a given fundamental triplet.

The next model we consider, also from Kepler's list, consists of truncated paraboloids placed back to back. Here

$$f(x) = \sqrt{\frac{aR^2 - (R^2 - r^2)|x|}{a}},$$

so that, from (4), we obtain

$$V_2 = \pi l \frac{R^2 + r^2}{2}. \tag{6}$$

As in the model leading to V_1 there is a cusp at $x = 0$, rather than a smooth turning point, which suggests that V_2, just like V_1, will underestimate the volume of a real barrel with the same fundamental triplet, although obviously $V_2 > V_1$. Notice that with the paraboloid, the curvature of the profile increases towards the ends of the curve. Finally, notice that (6) is just the average of volumes of two cylinders which provide upper and lower bounds for the volume of the barrel.

The next two models, also from Kepler, lead to overestimation of the volume.

The first of these, known as the *spheroid*, is in fact an ellipsoid, truncated at both ends of its long axis. It was already known in the 14th century, and was associated with the name of Jean de Murs. Here we have

$$f(x) = \sqrt{\frac{a^2R^2 - (R^2 - r^2)x^2}{a^2}},$$

giving

$$V_8 = \pi l \frac{2R^2 + r^2}{3}. \tag{7}$$

Although as simple in form as V_2 and at first sight a good approximation to the volume of a barrel, its suffix indicates that it is the largest of our eight V_is.

A smaller volume is obtained with Kepler's *parabolic spindle* arising from the quadratic

$$f(x) = R - \frac{(R - r)x^2}{a^2},$$

which gives

$$V_7 = \pi l \frac{8R^2 + 4Rr + 3r^2}{15}, \tag{8}$$

and is still used by some trusting gaugers.

The first satisfactory explanation for the values of V_7 and V_8 to be overestimates of the real volume was given in 1741 by Camus [1]. He pointed out that the bending of the staves forming the sides of the barrel, under the combined action of water and fire at the hands of coopers, is most pronounced at the middle of the staves, and negligible at the ends. Thus, the curvature of a function f giving a good model should be large at the bunghole, decrease as x increases from 0 to a, and vanish when $x = a$. With the parabolic spindle for example, the curvature does decrease from $x = 0$ to $x = a$, but too slowly, since it does not vanish at a. Camus consequently modified the parabolic spindle construction, with a more complex f using a combination of a parabola to the right of $x = 0$ and then a tangent to it, which passes through B, so the curvature becomes zero abruptly, while the slope $f'(x)$ is nevertheless continuous. The volume is then reduced to

$$V_3 = \pi l \frac{64R^2 + 37Rr + 34r^2}{135}. \tag{9}$$

We may check from (3) that $V_2 < V_3 < V_7$ by taking into account that $r < R \leq 2r$ for all real barrels. Camus' model represented real progress, providing a value for V_3 between the underestimates, V_1 and V_2, and the overestimates, V_7 and V_8.

3. New geometric models

As far as we know, no satisfactory geometric model has been proposed in France since Camus, although a number of 'empirical models' were introduced which are examples of the general structure (2). It is probable that Camus had only an intuitive understanding of curvature, immersed as theoretical gaugers were in the Kepler tradition of straight line segments and conic sections (which never have curvature zero). The condition that curvature should decrease to zero is not easy to handle in a strictly geometric framework, but becomes tractable if we use the second derivative $f''(x)$ in the classical value of the curvature at x as follows:

$$\frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}.$$

In any case, Camus' concept leads to a function on $[0, a]$ decreasing from $f(0) = R$ to $f(a) = r$, subject to the conditions $f'(0+) = 0$ and $f''(0+) < 0$, with the curvature steadily decreasing on $[0, a]$ and $f''(a) = 0$. Accordingly, we propose three simple models satisfying these conditions.

The first begins with the cubic polynomial on $[0, a]$ specified by

$$f(x) = \frac{R-r}{2a^3}x^3 - 3\frac{R-r}{2a^2}x^2 + R,$$

which results in the volume

$$V_4 = \pi l \frac{68R^2 + 39Rr + 33r^2}{140}, \tag{10}$$

which is easily seen to satisfy $V_4 > V_3$ by using (3).

Our next function is trigonometric

$$f(x) = (R-r)\cos\frac{\pi x}{2a} + r.$$

The usual conditions including those of Camus are again satisfied, and we are led to the volume

$$V_5 = 2\int_0^a \left[(R-r)\cos\frac{\pi x}{2a} + r\right]^2 \mathrm{d}x = \pi l\left[\frac{R^2}{2} + \left(\frac{4}{\pi} - 1\right)Rr + \left(\frac{3}{2} - \frac{4}{\pi}\right)r^2\right], \tag{11}$$

with the help of the formula $1 + \cos 2\theta = 2\cos^2\theta$. A further step reveals that $V_4 < V_5$ for all usual triplets.

Finally, another smooth model is generated by the biquadratic (quadratic in x^2) function

$$f(x) = \frac{(R-r)x^4}{5a^4} - \frac{6(R-r)x^2}{5a^2} + R.$$

Calculation of $\int_0^a f^2\,\mathrm{d}x$ is somewhat cumbersome, as is the final result:

$$V_6 = \pi l \frac{3968R^2 + 2144Rr + 1763r^2}{7875}. \tag{12}$$

We have that V_6 is a little larger than V_5, but smaller than V_7, whatever the fundamental triplet.

Having tried with colleagues in 1998–1999 to raise the interest of pupils in applied mathematics, the Belgian mathematician G. Noël wrote to the cooper Radoux in Jonzac, asking for information on his gauging technique. A short answer described the profile of a stave as the graph of a biquadratic polynomial! At about this time, addressing an inquiry from a one-time collaborator, Germain Bonte, the author had also envisaged such a model, and was thus led to revive geometric modelling.

To conclude this section, note that V_4, V_5, and V_6 satisfy

$$V_1 < V_2 < V_3 < V_4 < V_5 < V_6 < V_7 < V_8; \tag{13}$$

the newcomers thus sitting between the underestimates, V_1 and V_2, and the overestimates, V_7 and V_8, and indeed doing a little better than the slight underestimate of Camus, V_3.

Thus, any one of V_4, V_5, or V_6 could be used as an 'honest' estimate of the true volume, the more so since they are very close to each other as the following numerical examples demonstrate. With the triplet $(l, D, d) = (12, 10, 8)$ which has swelling 1.25, we find the following sequence of V_is rounded to whole litres: 767, 773, 805, 810, 814, 815, 824, 829, in the same order as (13).

The 'official' measure (1) gives $W_1 = 779$ for the above triplet, which is something of an underestimate. Of course, W_1 may be accurate when applied to barrels for which it was designed.

If we take a more modest swelling, 1.125, using the triplet $(l, D, d) = (11, 9, 8)$ we obtain $W_1 = 623$, which is even less than $V_1 = 625$. Here, $V_2 = 626$, $V_3 = 641$, $V_4 = 644$, $V_5 = 645$, $V_6 = 646$, $V_7 = 650$, and $V_8 = 651$.

In all our calculations we have used π to several decimal places, although a gauger may well still use the traditional approximation $\frac{22}{7}$ on a simple calculator.

4. Empirical expressions

Camus' geometrically obtained expression (9) for the volume V_3 was changed empirically by Dez in 1773 and 1785 (see [2] and [3]) into

$$W_3 = \frac{\pi l}{4}\left[\frac{D+d}{2} + \frac{1}{8}(D-d)\right]^2 = \pi l\frac{(5R+3r)^2}{64} = \pi l\frac{25R^2+30Rr+9r^2}{64}, \tag{14}$$

using again a traditional weighted mean of R^2, Rr, r^2.

It turns out by using (14) that $W_3 < V_4$, but if the swelling $\tau = R/r < 1.333$ then $W_3 > V_3$, so under this condition it tends to give a good approximation to the real volume. (Dez made an error in his calculation of the difference $V_3 - W_3$, reaching an over-optimistic conclusion.)

During the French Revolution, another empirical formula was devised, which was suggested, as far as the author can determine (the original text is lost), by a formal edict (*Instruction ministérielle de pluviôse de l'an VII*). This date is almost equivalent to February 1799. In this case,

$$W_7 = \pi l\frac{(2D+d)^2}{36} = \pi l\frac{4R^2+4Rr+r^2}{9}, \tag{15}$$

containing again a weighted mean of R^2, Rr, r^2. Note using (15) that $W_7 < V_7$ as noticed in [6], although comparison with our V_6 shows that $V_6 < W_7$.

The formulae for W_3 and W_7 may be deemed of acceptable quality, but unlike the formulae obtained from geometric models they cannot be extended to allow for satisfactory gauging of partially filled barrels, a delicate problem which we do not deal with here. Instead, we direct the interested reader to Meskens [5].

Appendix A. (By E. Seneta.)

Notice that each of the 'weighted mean' volume formulae (5)–(12) is given by an expression of the following form:

$$V = 2\pi r^2 \Psi(\tau), \tag{16}$$

where $\tau = R/r$. Moreover, for $y \geq 1$,

$$\Psi(y) = \int_0^a \psi\left(\frac{x}{a}, y\right) \mathrm{d}x, \tag{17}$$

where $a > 0$ is a constant and $\psi(w, y)$, $(w, y) \in (0, 1) \times [1, \infty)$, is a nonnegative function which is a quadratic in y for fixed w. Thus, changing the variable of integration in (17) to $w = x/a$, and by putting $l = 2a$, from (16) and (17) we obtain

$$V = \pi l r^2 \int_0^1 \psi(w, \tau) \, \mathrm{d}w. \tag{18}$$

Consequently, $\int_0^1 \psi(w, \tau)\,\mathrm{d}w$ is a quadratic in τ. Its coefficients must sum to unity since $\int_0^1 \psi(w, 1)\,\mathrm{d}w = 1$. This follows from (18) since, when $\tau = 1$, V is the volume $\pi l r^2$ of a cylinder of radius $r = R$ and length l.

References

[1] Camus, C. (1741). Sur un instrument propre à jauger les tonneaux et les autres vaisseaux qui servent à contenir les liqueurs. *Histoire Acad. R. Sci. Paris, avec Mémoires Math. Phys. même Année,* pp. 385–402.

[2] Dez, M. (1773). Mémoire sur la théorie du jaugeage. *Mém. Math. Phys. prés. Acad. R. Sci. Savans, lûs Assem.,* pp. 383–389.

[3] Dez, M. (1785). Entries 'jauge' and 'jaugeage'. In *Encyclopédie Méthodique de Panckoucke-Plomteux,* Paris-Liège, Vol. II, pp. 246–257.

[4] Kepler, J. (1960). *Gesammelte Werke,* Vol. IX. Beck, München.

[5] Meskens, A. (1994). Wine gauging in late 16th and early 17th century Antwerp. *Historia Mathematica* **21,** 121–147.

[6] Péraux, P. (1887). Table de jaugeage pour les tonneaux. *La Nature* **15,** 402–403.