Reading group on quantum groups and affine Lie algebras

University of Oxford, Hilary Term 2016

Organisers: Emily Cliff, André Henriques, Kobi Kremnizer.

Week 1: 21 January **André, Kobi, Emily**

• Introduction and overview by André. Additional wisdom from Kobi.
• Outline of plan for each talk, solicitation of volunteers by Emily/André.

Week 2: 28 January **Manuel**

Goal: state and prove the equivalence between (topological) modular functors and modular tensor categories. (1 hour)

- Definitions of modular functors and modular tensor categories.
- Note: we need an extra condition on the modular functor to ensure rigidity in the corresponding tensor category.
- Restricting the modular functor to genus zero gives a rigid braided tensor category. Higher genera give modularity.
- References: Chapter 5 of [BK01], Bruce Bartlett.

Jack

Goal: explain the Riemann-Hilbert correspondence. (1 hour)

- Define D-modules, pushforward and pullback functors.
- References: Section 6.3 of [BK01] ...

Week 3: 4 February **Thomas**

Goal: explain the equivalence of topological and complex modular functors. (1 hour)

- Define complex modular functors (requires defining the moduli space of curves first).
- References: Chapter 6 of [BK01].

Adam

Goal: introduce the theory of integrable representations of affine Lie algebras. (1 hour)

- Define affine Lie algebras. Define and classify integrable representations.
- References: Section 7.1 of [BK01] ...

Week 4: 11 February **Craig**

Goal: introduce quantum groups and their representations. (2 hours)

- Introduce quantum groups via formulas and integral forms.
- Introduce Verma modules, integrable modules. Define Weyl modules (i.e. standard modules) and dual Weyl modules (i.e. costandard modules) using induction functors.
- Begin with generic q, but eventually we're interested in q a root of unity.
- References: [APW91], [Saw06], notes of Agustín García Iglesias ("Representations of quantum groups at roots of unity").

Week 5: 18 February **Robert**

Goal: introduce R-matrices using the theory of doubles, and give the specific example of the positive part of the quantum group. (2 hours)

- Begin abstractly with discussion of pointed Hopf algebras, doubles, R-matrices.
- Our main example: the positive part of the quantum group. Introduce inner product $\langle \cdot, \cdot \rangle$.
- Braided tensor structure on the category of representations of the quantum group.
- References: For quantum groups: [CP95], [ES02], the beginning of [Lus10]. For pointed Hopf algebras: [AS02].
- Maybe look at section 37 of [KL94] for consistency with notation.

Week 6: 25 February **Lena**

Goal: define the category of tilting modules modulo negligible morphisms and prove it is semi-simple . (2 hours)

- Define tilting modules, and prove that the tensor product of two tilting modules is a tilting module. Or more generally, prove that the tensor product of two modules which admit a standard filtration again admits a standard filtration.
- Introduce negligible morphisms, and prove that the quotient category of tilting modules modulo negligibles is semisimple. Classify its simple objects.
- References: Notes by Feldvoss ("Tilting modules for quantum groups"). [And92]. **[Saw06]**.

Weeks 7 and 8: 3 and 10 March **Emily and Marina**

Goal: construct the complex modular functor (and all its structure) from the category of representations of an affine Lie algebra. (4 hours)

- The speaker(s) should cover as much as possible, and leave us excited to return to this topic in greater detail next term.
- References: Chapter 7 of [BK01] (it's important to use the book, not the online version—you can borrow a copy from André). [AU07] as a secondary resource. André.

References

[And92] Henning Haahr Andersen. Tensor products of quantized tilting modules. *Comm. Math. Phys.*, 149(1):149–159, 1992.

[APW91] Henning Haahr Andersen, Patrick Polo, and Ke Xin Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991.

[AS02] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.

[AU07] Jørgen Ellegaard Andersen and Kenji Ueno. Geometric construction of modular functors from conformal field theory. *J. Knot Theory Ramifications*, 16(2):127–202, 2007.

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[CP95] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original.

[ES02] Pavel Etingof and Olivier Schiffmann. *Lectures on quantum groups*. Lectures in Mathematical Physics. International Press, Somerville, MA, second edition, 2002.

[KL94] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. IV. *J. Amer. Math. Soc.*, 7(2):383–453, 1994.

[Lus10] George Lusztig. *Introduction to quantum groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[Saw06] Stephen F. Sawin. Quantum groups at roots of unity and modularity. *J. Knot Theory Ramifications*, 15(10):1245–1277, 2006.