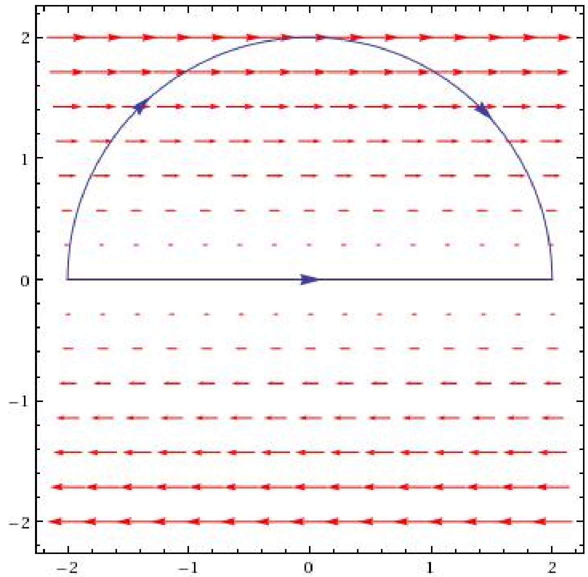
1. Consider the vector field $\mathbf{F} = (y, 0)$ on $\mathbb{R}^2$.

   (a) Draw a sketch of $\mathbf{F}$ on the region where $-2 \le x \le 2$ and $-2 \le y \le 2$. Check you answer with the instructor.

   SOLUTION:

   Below is the image for parts (a) and (b)

   (b) Consider the following two curves which *start* at $A = (-2, 0)$ and *end* at $B = (2, 0)$, namely the line segment C_1 and upper semicircle C_2.

   Add these curves to your sketch, and compute both $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Check you answers with the instructor.

   SOLUTION:

   Parametrize C_1 by $\mathbf{r}_1(t) = (t, 0), -2 \le t \le 2$ and parametrize C_2 by $\mathbf{r}_2(t) = (-2\cos t, 2\sin t), 0 \le t \le \pi$. We have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 F(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\, dt = \int_0^2 (0,0) \cdot (1,0)\, dt = 0$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi F(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)\, dt = \int_0^\pi (2\sin t, 0) \cdot (2\sin t, 2\cos t)\, dt = 4\int_0^\pi \sin^2(t)\, dt$$

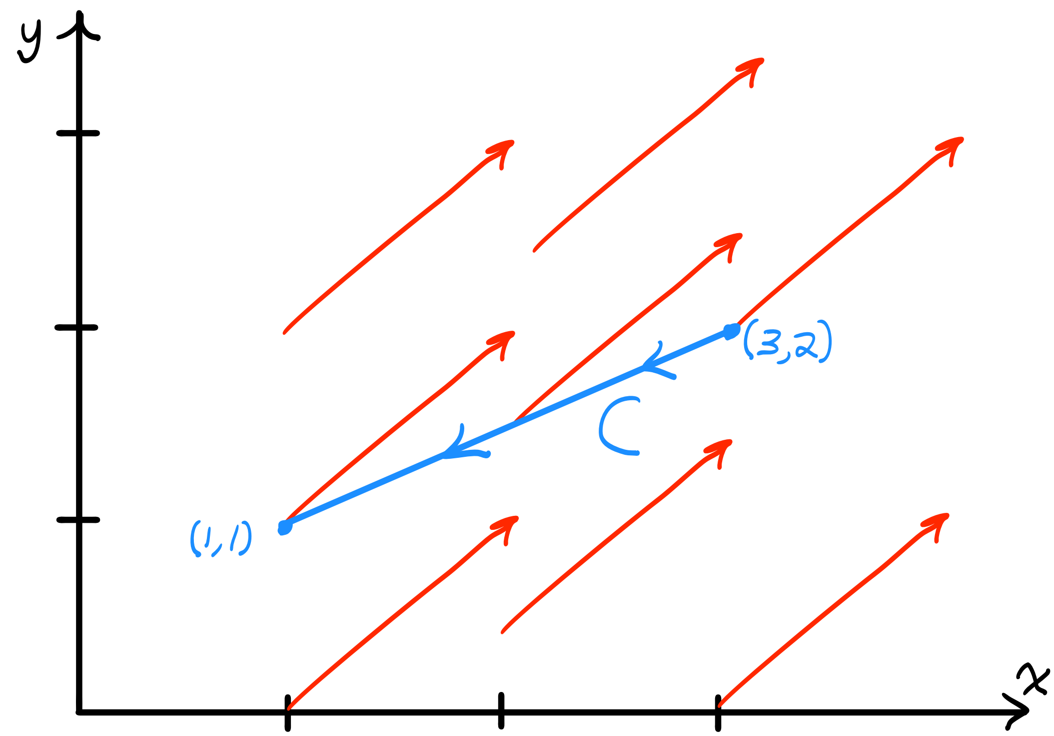
$$= 4 \cdot \frac{1}{2}\left[t - \frac{1}{2}\sin(2t)\right]_0^\pi = 2\pi$$

   (c) Based on your answer in (b), could $\mathbf{F}$ be ∇f for some $f\colon \mathbb{R}^2 \to \mathbb{R}$? Explain why or why not.

   SOLUTION:

   By the Fundamental Theorem of Line Integrals, if $\mathbf{F} = \nabla f$ for some $f\colon \mathbb{R}^2 \to \mathbb{R}$ then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent for any curve C starting at $A = (-2, 0)$ and ending at $B = (2, 0)$. Since we obtained different answers for the paths C_1 and C_2, $\mathbf{F}$ cannot be of this form.

2. Consider the curve C and vector field $\mathbf{F}$ shown below.



(a) Calculate $\mathbf{F}\cdot\mathbf{T}$, where here $\mathbf{T}$ is the unit tangent vector along C. Without parameterizing C, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ by using the fact that it is equal to $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

SOLUTION:

From the picture we suppose that $\mathbf{F}(x,y) = (1,1)$. We have $\mathbf{T} = \frac{1}{\sqrt{5}}(-2,-1)$, so $\mathbf{F}\cdot\mathbf{T} = \frac{-3}{\sqrt{5}}$. So

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \frac{-3}{\sqrt{5}}\int_C ds = -3$$

since $\int_C ds$ is simply the distance between $(1,1)$ and $(3,2)$.

(b) Find a parameterization of C and a formula for $\mathbf{F}$. Use them to check your answer in (a) by computing $\int_C \mathbf{F}\cdot d\mathbf{r}$ explicitly.

SOLUTION:

Parametrize C by $\mathbf{r}(t) = (3-2t, 2-t), 0 \le t \le 1$. We already have $\mathbf{F} = (1,1)$. So

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (1,1)\cdot(-2,-1)\,dt = -3$$

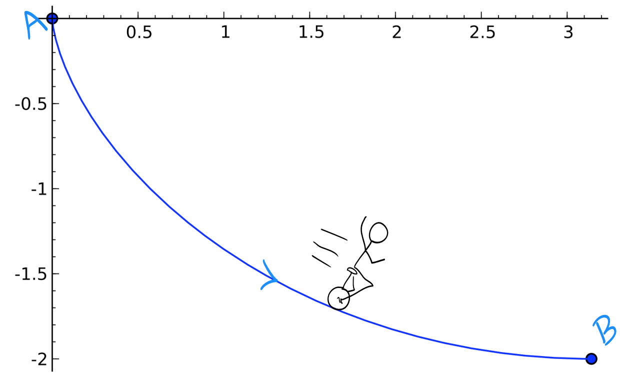
3. Consider the points $A = (0,0)$ and $B = (\pi,-2)$. Suppose an object of mass m moves from A to B and experiences the constant force $\mathbf{F} = -mg\mathbf{j}$, where g is the gravitational constant.

(a) If the object follows the straight line from A to B, calculate the work W done by gravity using the formula from the first week of class.

SOLUTION:

Recall that the work done on an object moving along a straight line subject to a constant force $\mathbf{F}$ is $W = \mathbf{F}\cdot\mathbf{D}$, where $\mathbf{D}$ is the displacement vector. In this case $\mathbf{D} = (\pi,-2)$ and $\mathbf{F} = (0,-mg)$. So $W = (\pi,-2)\cdot(0,-mg) = 2mg$.

(b) Now suppose the object follows half of an inverted cycloid C as shown below. Explicitly parameterize C and use that to calculate the work done via a line integral.



SOLUTION:

A parametrization for the inverted cycloid C is $\mathbf{r}(t) = (t - \sin t, \cos t - 1), 0 \le t \le \pi$. So

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (0, -mg) \cdot (1 - \cos t, -\sin t)\, dt = \int_0^\pi mg \sin t\, dt = mg\,[-\cos t]_0^\pi = 2mg$$

(c) Find a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla f = \mathbf{F}$. Use the Fundamental Theorem of Line Integrals to check your answers for (a) and (b). Have you seen the quantity $-f$ anywhere before? If so, what was its name?

SOLUTION:

If such an f exists, we must have $f_x = 0$ and $f_y = -mg$. Integrating $-mg$ with respect to y we obtain $f = -mgy + C(x)$, where $C(x)$ is some function of x. Differentiating this with respect to x we obtain $f_x = C'(x) = 0$, so $f = -mgy + K$, where K is a constant, is a potential function for $\mathbf{F}$.

By the Fundamental Theorem of Line integrals, both (a) and (b) must have the same answer, namely

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L \nabla(f) \cdot d\mathbf{r} = f(B) - f(A) = f(\pi, -2) - f(0,0) = (-mg(-2) + K) - K = 2mg$$

where L is the line segment from A to B and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla(f) \cdot d\mathbf{r} = f(B) - f(A) = 2mg$$

The quantity $-f$ is called the *potential energy*.

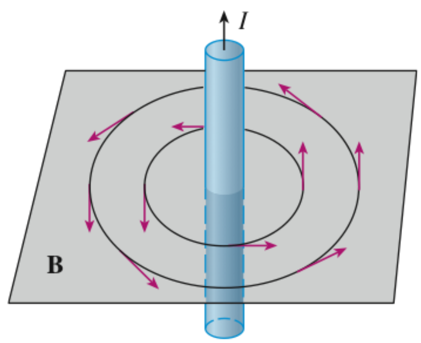
4. If you get this far, work #52 from Section 16.2:

48. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C, and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



SOLUTION:

We are assuming that $\mathbf{B}$ has magnitude which only depends on the distance from the wire. So $B = |\mathbf{B}|$ is constant along any circle centered around the wire in a plane perpendicular to the wire. Let $C = \mathbf{r}(t)$ be such a circle with radius r parametrized in the counterclockwise direction and let B denote the magnitude of $\mathbf{B}$ along C. Note that $\mathbf{B}(\mathbf{r}(t))$ is a positive multiple of $\mathbf{r}'(t)$ by definition. So it follows that $\mathbf{T}(t)$, the unit tangent vector to C, is given by $\mathbf{T}(t) = \dfrac{\mathbf{B}(\mathbf{r}(t))}{B}$. We have

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_C \mathbf{B} \cdot \mathbf{T} ds = \int_C \frac{\mathbf{B} \cdot \mathbf{B}}{B} ds = B \int_C ds = 2\pi r B$$

By Ampere's Law, $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$, so we have $2\pi r B = \mu_0 I$, or $B = \dfrac{\mu_0 I}{2\pi r}$.