Thursday, February 28 ∗∗ *Curves and integration.*

1. Consider the curve C in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = \left(e^t \cos t\right)\mathbf{i} + 2\mathbf{j} + \left(e^t \sin t\right)\mathbf{k}$$

   (a) Draw a sketch of C.

   (b) Calculate the arc length function $s(t)$, which gives the length of the segment of C between $\mathbf{r}(0)$ and $\mathbf{r}(t)$ as a function of the time t for all $t \geq 0$. Check your answer with the instructor.

   (c) Now invert this function to find the inverse function $t(s)$. This gives time as a function of arclength, that is, tells how long you must travel to go a certain distance.

   (d) Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is a function. We can get another parameterization of C by considering the composition

$$\mathbf{f}(s) = \mathbf{r}\big(h(s)\big)$$

   This is called a *reparametrization.* Find a choice of h so that

      i. $\mathbf{f}(0) = \mathbf{r}(0)$

      ii. The length of the segment of C between $\mathbf{f}(0)$ and $\mathbf{f}(s)$ is s. (This is called parametrizing by arc length.)

   Check your answer with the instructor.

   (e) Without calculating anything, what is $|\mathbf{f}'(s)|$?

2. Consider the curve C given by the parametrization $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$ where $\mathbf{r}(t) = (\sin t, \cos t, \sin^2 t)$.

   (a) Show that C is in the intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

   (b) Use (a) to help you sketch the curve C.

3. (a) Sketch the top half of the sphere $x^2 + y^2 + z^2 = 5$. Check that $P = \left(1, 1, \sqrt{3}\right)$ is on this sphere and add this point to your picture.

   (b) Find a function $f(x, y)$ whose graph is the top-half of the sphere. Hint: solve for z.

   (c) Imagine an ant walking along the surface of the sphere. It walks *down* the sphere along the path C that passes through the point P in the direction parallel to the yz-plane. Draw this path in your picture.

   (d) Find a parametrization $\mathbf{r}(t)$ of the ant's path along the portion of the sphere shown in your picture. Specify the domain for $\mathbf{r}$, i.e. the initial time when the ant is at P and the final time when it hits the xy-plane.

4. As in 1(d), consider a reparametrization

$$\mathbf{f}(s) = \mathbf{r}\big(h(s)\big)$$

   of an arbitrary vector-valued function $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$. Use the chain rule to calculate $|\mathbf{f}'(s)|$ in terms of $\mathbf{r}'$ and h'.