Thursday, January 31 $*$ **Solutions** $*$ *Functions of several variables; Limits.*

1. For each of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, draw a sketch of the graph together with pictures of some level sets.

   (a) $f(x,y) = xy$

   (b) $f(\mathbf{x}) = |\mathbf{x}|$. Please note here that $\mathbf{x}$ is a vector. In coordinates, this function is $f(x,y) = \sqrt{x^2+y^2}$.

   For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

   Solution.

   (a) The graph of the function $f(x,y) = xy$ is

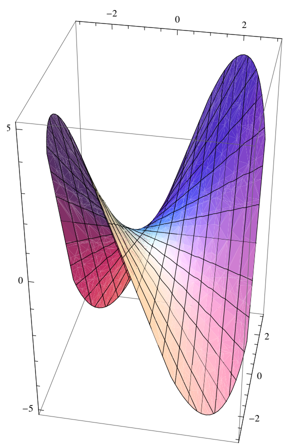

Figure 1: Graph of $f(x,y) = xy$.

The graph of the level sets $f(x,y) = -2,-1,0,1,2$ is

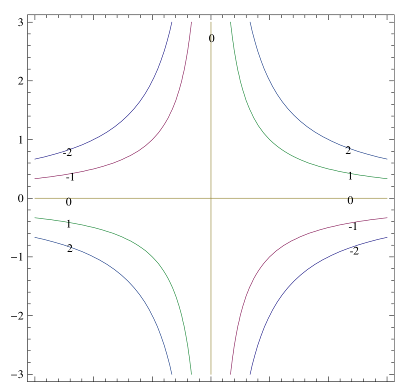

Figure 2: Graph of Level Sets of $f(x,y) = xy$.

The graph of $f(x, y) = xy$ is a hyperbolic paraboloid since the horizontal traces are hyperbolas and the vertical traces are parabolas.

(b) The graph of the function $f(\mathbf{x}) = |\mathbf{x}|$ is

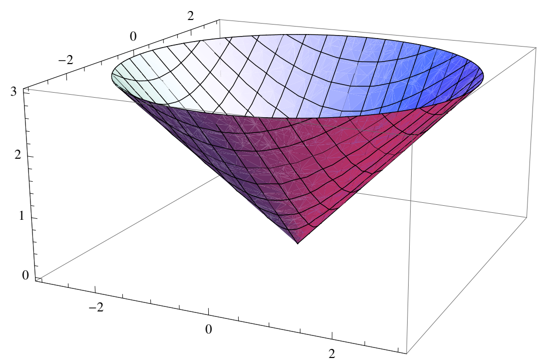

Figure 3: Graph of $f(\mathbf{x}) = |\mathbf{x}|$.

The graph of the level sets $f(x, y) = 0, 1, 2, 3$ is

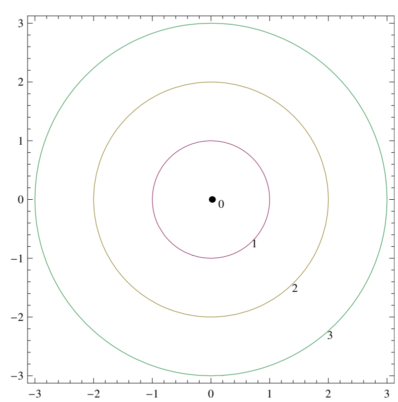

Figure 4: Graph of Level Sets of $f(\mathbf{x}) = |\mathbf{x}|$.

The graph of $f(\mathbf{x}) = |\mathbf{x}|$ is not a quadric surface because it cannot be written as $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$. It is the top half of a cone, which is a quadric surface.

2. Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{2x^3 y}{x^6 + y^2} \quad \text{for } (x, y) \neq \mathbf{0}$$

In this problem, you'll consider $\lim_{(x,y)\to\mathbf{0}} f(x, y)$.

(a) Look at the values of f on the x- and y-axes. What do these values show the limit $\lim_{(x,y)\to\mathbf{0}} f(x,y)$ must be **if it exists**?

Solution. Along $y=0$, $\displaystyle\lim_{(x,y)\to\mathbf{0}} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0}\frac{0}{x^6} = 0.$

Along $x=0$, $\displaystyle\lim_{(x,y)\to\mathbf{0}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0}\frac{0}{y^2} = 0.$

Thus, should it exist, we must have $\displaystyle\lim_{(x,y)\to\mathbf{0}} f(x,y) = 0.$

(b) Show that along each line in $\mathbb{R}^2$ through the origin, the limit of f exists and is 0.

Solution. Any line through the origin besides $x=0$ or $y=0$ can be written as $y=mx,\ m\neq 0$.

Along $y=mx$, $\displaystyle\lim_{(x,y)\to\mathbf{0}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0}\frac{2mx^4}{x^6+m^2x^2} = \lim_{x\to 0}\frac{2mx^2}{x^4+m^2} = 0.$

(c) Despite this, show that the limit $\lim_{(x,y)\to\mathbf{0}} f(x,y)$ does not exist by finding a curve over which f takes on the constant value 1.

Solution. Along $y=x^3$, $\displaystyle\lim_{(x,y)\to\mathbf{0}} f(x,y) = \lim_{x\to 0} f(x,x^3) = \lim_{x\to 0}\frac{2x^6}{x^6+x^6} = 1.$

3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \frac{xy^2}{\sqrt{x^2+y^2}} \quad \text{for } (x,y)\neq \mathbf{0}$$

In this problem, you'll show $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$.

(a) For $\epsilon = 1/2$, find some $\delta > 0$ so that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$. Hint: As with the example in class, the key is to relate $|x|$ and $|y|$ with $|\mathbf{h}|$.

Solution. Note that $|x|, |y| \le |\mathbf{h}|$. For $\epsilon = 1/2$, let $\delta = 1/\sqrt{2}$. Then $0 < |\mathbf{h}| < \delta$ implies

$$|f(\mathbf{h})| \le \frac{|\mathbf{h}|^3}{|\mathbf{h}|} = |\mathbf{h}|^2 < \delta^2 = \frac{1}{2}.$$

(b) Repeat with $\epsilon = 1/10$.

Solution. For $\epsilon = 1/10$, let $\delta = 1/\sqrt{10}$. Then $0 < |\mathbf{h}| < \delta$ implies

$$|f(\mathbf{h})| \le |\mathbf{h}|^2 < \delta^2 = \frac{1}{10}.$$

(c) Now show that $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$. That is, given an arbitrary $\epsilon > 0$, find a $\delta > 0$ so that that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$.

Solution. Given $\epsilon > 0$, let $\delta = \sqrt{\epsilon}$. Then $0 < |\mathbf{h}| < \delta$ implies

$$|f(\mathbf{h})| \leq |\mathbf{h}|^2 < \delta^2 = \epsilon.$$

(d) Explain why the limit laws that you learned in class on Wednesday aren't enough to compute this particular limit.

Solution. $f(x, y)$ cannot be written as $f(x, y) = g(x, y)h(x, y)$ so that $\lim\limits_{|\mathbf{x}|\to 0} g(\mathbf{x})$ and $\lim\limits_{|\mathbf{x}|\to 0} h(\mathbf{x})$ both exist and are easier to compute than $\lim\limits_{|\mathbf{x}|\to 0} f(\mathbf{x})$.