Tuesday, January 22 $*$ **Solutions** $*$ *Projections, distances, and planes.*

1. Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 1\mathbf{j}$

   (a) Calculate $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$ and draw a picture of it together with $\mathbf{a}$ and $\mathbf{b}$.

   SOLUTION:

   $\text{proj}_{\mathbf{b}}\mathbf{a} = \langle 2/5, -1/5\rangle$. This is drawn below (b).

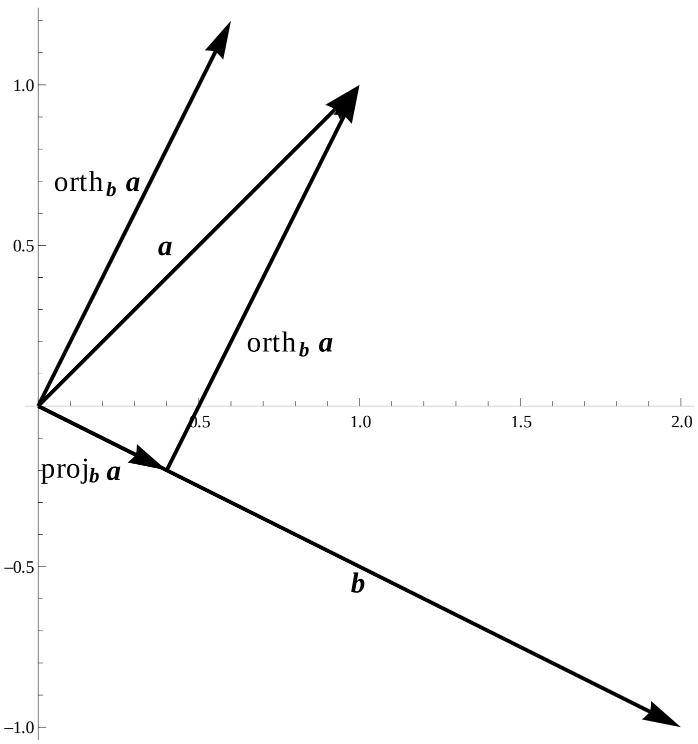
   (b) The orthogonal complement of $\mathbf{a}$ with respect to $\mathbf{b}$ is the vector

   $$\text{orth}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}.$$

   Find $\text{orth}_{\mathbf{b}}\mathbf{a}$ and $\text{orth}_{\mathbf{b}}\mathbf{a}$ and draw two copies of it in your picture from part (a), one based at $\mathbf{0}$ and the other at $\text{proj}_{\mathbf{b}}\mathbf{a}$.

   SOLUTION:

   $\text{orth}_{\mathbf{b}}\mathbf{a} = \langle 1, 1\rangle - \langle 2/5, -1/5\rangle = \langle 3/5, 6/5\rangle$



   (c) Check that $\text{orth}_{\mathbf{b}}(\mathbf{a})$ calculated in (b) is orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$ calculated in (a).

   SOLUTION:

   $\langle 2/5, -1/5\rangle \cdot \langle 3/5, 6/5\rangle = 6/25 - 6/25 = 0$, so $\text{orth}_{\mathbf{b}}(\mathbf{a})$ and $\text{proj}_{\mathbf{b}}\mathbf{a}$ are orthogonal.

   (d) Find the distance of the point $(1, 1)$ from the line $(x, y) = t(2, -1)$.

   SOLUTION:

   This is the length of $\text{orth}_{\mathbf{b}}(\mathbf{a})$, or $\sqrt{(3/5)^2 + (6/5)^2} = 3\sqrt{5}/5$.

2. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$. Use the definitions of $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\text{orth}_{\mathbf{b}}\mathbf{a}$ to show that $\text{orth}_{\mathbf{b}}\mathbf{a}$ is always orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$.

   SOLUTION:

   Since $\text{proj}_{\mathbf{b}}\mathbf{a}$ points in the same direction as $\mathbf{b}$, it is equivalent to show that $\mathbf{b}$ is orthogonal to $\text{orth}_{\mathbf{b}}\mathbf{a}$. We take the dot product:

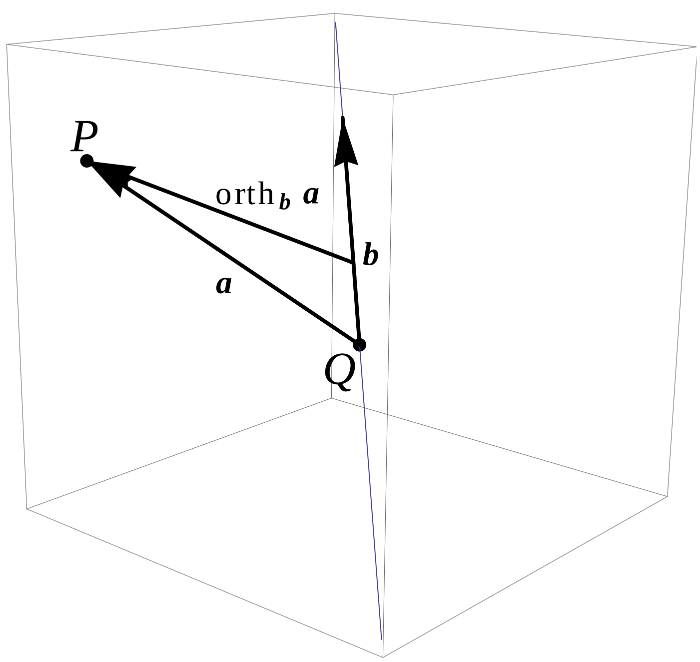
   $$\mathbf{b}\cdot \text{orth}_{\mathbf{b}}\mathbf{a} = \mathbf{b}\cdot\left(\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}\right) = \mathbf{b}\cdot\mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} = 0$$

   Since the dot product of $\mathbf{b}$ and $\text{orth}_{\mathbf{b}}\mathbf{a}$ is 0, they are orthogonal.

3. Find the distance between the point $P(3,4,-1)$ and the line $\mathbf{l}(t) = (2,3,-2) + t(1,-1,1)$.

   SOLUTION:

   Let $Q = (2,3,-2)$, $\mathbf{a} = \langle 3,4,-1\rangle - \langle 2,3,-2\rangle = \langle 1,1,1\rangle$ and $\mathbf{b} = \langle 1,-1,1\rangle$. The distance from P to $\mathbf{l}(t)$ is given by the magnitude of $\text{orth}_{\mathbf{b}}\mathbf{a}$ as shown below.



   $\text{proj}_{\mathbf{b}}\mathbf{a} = \langle 1/3, -1/3, 1/3\rangle$ and $\text{orth}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = \langle 2/3, 4/3, 2/3\rangle$. So the distance from P to $\mathbf{l}(t)$ is $|\text{orth}_{\mathbf{b}}\mathbf{a}| = \frac{2\sqrt{6}}{3}$.

4. Consider the equation of the plane $x + 2y + 3z = 12$.

   (a) Find a normal vector to the plane.

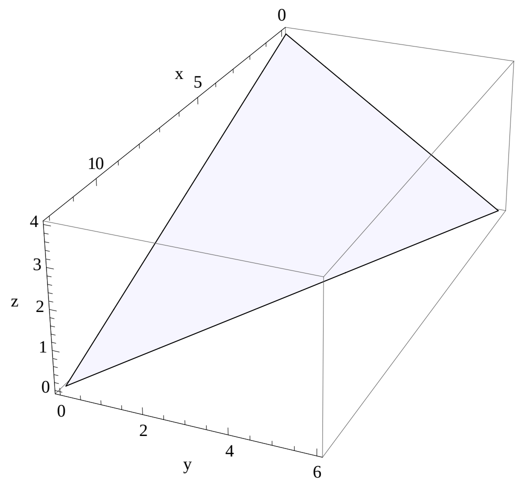
   SOLUTION:

   A normal vector is $n = \langle 1,2,3\rangle$.

(b) Find where the x, y, and z-axes intersect the plane. Sketch the portion of the plane in the first octant where $x \geq 0, y \geq 0, z \geq 0$.

SOLUTION:

The plane intersects the x, y, and z-axes respectively at $(12,0,0)$, $(0,6,0)$, and $(0,0,4)$. The sketch is shown below.



(c) Using the points in part (b), find two non-parallel vectors that are parallel to the plane.

SOLUTION:

The vectors $\mathbf{a} = \langle 12, 0, -4\rangle$ and $\mathbf{b} = \langle 0, 6, -4\rangle$ work. These vectors start at the intersection of the plane with the z-axis and end at the intersections with the x and y-axes respectively.

(d) Using the dot product to check that the vectors you found in (c) are really parallel to the plane.

SOLUTION:

A vector $\mathbf{v}$ is parallel to the plane if and only if it is orthogonal to a normal vector for the plane, that is $\mathbf{v} \cdot \mathbf{n} = 0$. So we check:

$$\mathbf{a} \cdot \mathbf{n} = \langle 12, 0, -4\rangle \cdot \langle 1, 2, 3\rangle = 12 + 0 - 12 = 0$$
$$\mathbf{b} \cdot \mathbf{n} = \langle 0, 6, -4\rangle \cdot \langle 1, 2, 3\rangle = 0 + 12 - 12 = 0$$

(e) Pick another normal vector $\mathbf{n}'$ to the plane and one of the points from (b). Use these to find an alternative equation for the plane. Compare this new equation to $x + 2y + 3z = 12$. How are these two equations related? Is it clear that they describe the same set of points (x, y, z) in $\mathbb{R}^3$?

SOLUTION:

We use the point $(0,0,4)$ and normal vector $\mathbf{n}' = 2\mathbf{n} = \langle 2, 4, 6\rangle$. The plane consists of all points (x, y, z) such that the vector $\langle x, y, z-4\rangle$ is orthogonal to the vector $\mathbf{n}'$. This is expressed by

$$\mathbf{n}' \cdot \langle x, y, z-4\rangle = 0$$

or

$$2x + 4y + 6(z-4) = 0 \quad \text{which is the same as} \quad 2x + 4y + 6z = 24.$$

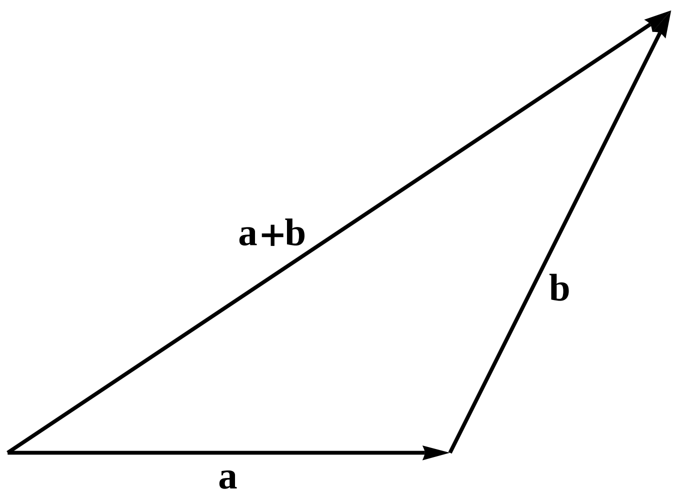
If we divide both sides by 2, we obtain the equation $x+2y+3z=12$, which is the original equation. These describe the same set of points because multiplying both sides of the original equation by any nonzero constant does not affect the solution set.

5. *The Triangle Inequality.* Let $\mathbf{a}$ and $\mathbf{b}$ be any vectors in $\mathbb{R}^n$. The triangle inequality states that $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}|+|\mathbf{b}|$.

   (a) Give a geometric interpretation of the triangle inequality.

   SOLUTION:

   Fit $\mathbf{a},\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ into a triangle as below. The triangle inequality says the sum of the lengths of the sides of the triangle corresponding to $\mathbf{a}$ and $\mathbf{b}$ is less than the length of the side corresponding to $\mathbf{a}+\mathbf{b}$.



   (b) Use what we know about the dot product to explain why $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. This is called the Cauchy-Schwartz inequality.

   SOLUTION:

   $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. So

$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \leq |\mathbf{a}||\mathbf{b}|, \text{ since } |\cos\theta| \leq 1.$$

   (c) Use part (b) to justify the triangle inequality.

   SOLUTION:

   It is equivalent to show

$$|\mathbf{a}+\mathbf{b}|^2 \leq (|\mathbf{a}|+|\mathbf{b}|)^2 = |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2$$

   We begin with the equality $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$. Since the dot product is distributive,

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = &\quad \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} \\ = &\quad |\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2 \\ \leq &\quad |\mathbf{a}|^2+2|\mathbf{a}||\mathbf{b}|+|\mathbf{b}|^2\end{aligned}$$

   where the last inequality follows from part (b). So this justifies the triangle inequality.