Recall: A vector field $\vec{F}$ is conservative if there is a function f s.t. $\nabla f = \vec{F}$.

$\boxed{1}$ Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$ and suppose $\vec{F}$ is a vector field on D with $P_y = Q_x$.

Is $\vec{F}$ conservative? → solution on slides.

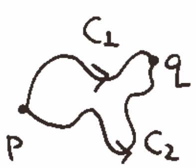
*Announcements.

Results in any dimension.

Assumption: For today, all vector fields have continuous first order partial derivatives.

Theorem A: $\vec{F}$ is conservative $\iff \int_C \vec{F}\cdot d\vec{r}$ is path independent

$\iff \int_C \vec{F}\cdot d\vec{r} = 0$ for all closed paths C.



$\left[\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} \right].$

path independence.

Method B: If we can find the potential f by hand, then $\vec{F}$ must be conservative.

Example: $\vec{F}(x,y,z) = \langle y^2 z,\ 2xyz - \sin y \sin z,\ xy^2 + \cos y \cos z \rangle$

- want f with $\nabla f = \vec{F}$
- $f_x = y^2 z \Rightarrow f(x,y,z) = xy^2 z + g(y,z)$.

  ↳ so $f_y(x,y,z) = 2xyz + g_y(y,z)$.
- but we want $f_y = 2xyz - \sin y \sin z$.

  $\Rightarrow g_y(y,z) = -\sin y \sin z$

  $\Rightarrow g(y,z) = \cos y \sin z + h(z)$.

  ↳ so $f(x,y,z) = xy^2 z + \cos y \sin z + h(z)$.
- ~~but we want f_z ...~~

  $\Rightarrow f_z(x,y,z) = xy^2 + \cos y \cos z + h'(z)$.
- but we want $f_z = xy^2 + \cos y \cos z$

  so $h'(z) = 0$, and we can take $h(z) = 0$

  $f(x,y,z) = xy^2 z + \cos y \sin z$.

* Always double check!

Results in $\mathbb{R}^2$

↳ Suppose $\vec{F} = \langle P, Q \rangle$ on $D \subset \mathbb{R}^2$.

Theorem C2 If $\vec{F}$ is conservative then $P_y = Q_x$

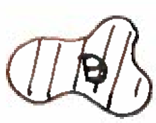
Theorem D2 If $D \subset \mathbb{R}^2$ is simply connected and $P_y - Q_x = 0$, then $\vec{F}$ is conservative.

Recall the proof of Theorem D2:

- By theorem A, it's enough to prove that $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed path C
- Step 1. We use Green's theorem to prove that $\int_{C'} \vec{F} \cdot d\vec{r} = 0$ for any simple closed path C'.
- Step 2: We show that any closed path can be split into a union of simple closed paths: $C = C_1 \cup C_2 \cup \cdots$

↳ So
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \cdots$$
$$= 0 + 0 + \cdots$$
$$= 0$$
☺.

Remarks on Step 1:



C'

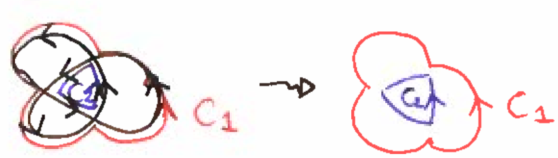
- we can fill in C' to get a region (*) $B \subset \mathbb{R}^2$ with $\partial B = \pm C'$

↑depends on orientation of C'.

- since D is simply connected, $B \subset D$ (**).

- Now by Green's Theorem (***)
$$\int_{C'} \vec{F} \cdot d\vec{r} = \pm \iint_B P_y - Q_x \, dA = \pm \iint_B 0 \, dA = 0.$$

Remarks on Step 2:



Results in $\mathbb{R}^3$ $\rightarrow$ $\vec{F} = \langle P, Q, R \rangle$ on $D \subset \mathbb{R}^3$.

Theorem C3 If $\vec{F}$ is conservative, then $\text{curl}\,\vec{F} = \langle 0,0,0 \rangle$

Theorem D3 If $D = \mathbb{R}^3$ ~~and~~ and $\text{curl}\,\vec{F} = \langle 0,0,0 \rangle$

then $\vec{F}$ is conservative.

Let's prove theorem D3: (compare to proof of Theorem D2)

* It's enough to show $\int_C \vec{F} \cdot d\vec{r} = 0$ for all ~~simple~~ closed paths. C

* Step 1. We use Stokes' Theorem to show $\int_{C'} \vec{F} \cdot d\vec{r} = 0$ for all simple closed paths C'

* Step 2. We show that any closed path $C \subset \mathbb{R}^3$ is a union of simple closed paths $C_1 \cup C_2 \cup \cdots$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \cdots$$

$$= 0 + 0 + \cdots \quad \text{by Step 1}$$

$$= 0. \quad \text{as required} \quad \square$$

$\hookrightarrow$ Step 2 is the same as in the 2d case, so we only need to work on Step 1.

- Let C' be a simple closed path in $\mathbb{R}^3$

Claim: There is an oriented surface S in $\mathbb{R}^3$ with $\partial S = C'$ (*), (**)

Then by Stokes' theorem (***)

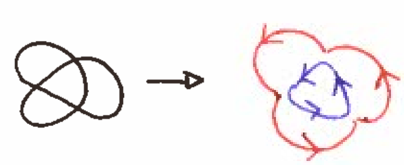
$$\int_{C'} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_S 0 \cdot d\vec{S} = 0.$$

So we just need to prove the claim:

- Let C' be a simple closed path in $\mathbb{R}^3$ (a "knot")

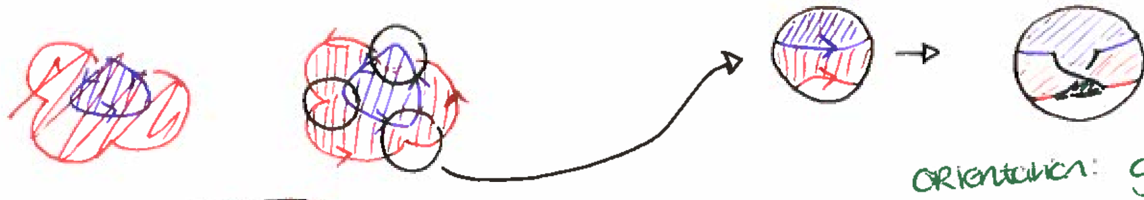
e.g. trefoil knot

$\longrightarrow$ flatten and divide into simple closed curves



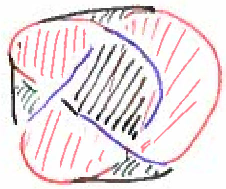
$\rightarrow$ lift the curves to be at different heights, and fill them in

Now any time there used to be a crossing, connect the two surfaces with a twisted strip:

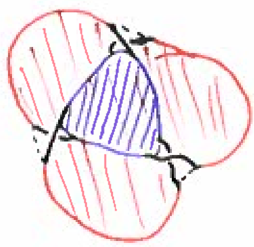


orientation: green = underside of red + purple surfaces.

red and purple – positive.



→ you get a surface with boundary the original curve



(purple disk is closer to you than the green disk).

- Review about irrotational & incompressible vector fields. [see slides]

↳ Suppose you don't know anything about $D \subset \mathbb{R}^2$

but I tell you there is a vector field $\vec{F} = \langle P, Q \rangle$

with $P_y - Q_x = 0$, but which is not conservative.

$\boxed{Q}$ What can you say about D?