Last time: linear change of coordinates

Recall that for a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ with Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & u_v \end{vmatrix}$$

we have the following formula:

$$\iint_{T(D)} f(x,y)dA = \iint_D f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

Use this to calculate the area of the ellipse $B = \{\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \leq 1\}$, by finding a linear transformation T with $T(D) = B$, where B is the unit disk $\{u^2 + v^2 \leq 1\}$.

- a I don't know what to do.
- b I found T, but now I don't know what to do.
- c I found T and the Jacobian, but I'm stuck now.
- d I'm done.

Solution

To find T we need to specify the values of $T(1,0)$ and $T(0,1)$. Since $(1,0)$ and $(0,1)$ are both on the boundary of D, we want $T(1,0)$ and $T(0,1)$ to be on the boundary of B.

$$T(1,0) = (\alpha, 0); \qquad T(0,1) = (0, \beta).$$

So $T(u,v) = (\alpha u, \beta v)$.

Now we need to substitute $T(u,v)$ into the function we're integrating, but that's $f = 1$, so we'll still have the constant function 1.

Solution

Next we need to find the Jacobian of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & u_v \end{vmatrix} = \begin{vmatrix} \alpha & 0 \\ 0 & \beta \end{vmatrix} = \alpha\beta > 0.$$

So

$$\text{Area}(B) = \iint_{T(D)} dA = \iint_D \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA = \iint_D \alpha\beta dA = \alpha\beta \text{Area}(D) = \alpha\beta\pi.$$

Practice with image and one-to-one

Let $T(r,\theta) = (r\cos\theta, r\sin\theta)$.
Let $D = [0,\infty) \times [0,2\pi)$. Is the image of T all of $\mathbb{R}^2$? Is T one-to-one on D?

(a) Yes and yes.

(b) Yes and no.

(c) No and yes.

(d) No and no.

(e) I don't know.

$T(D) = \mathbb{R}^2$
T is not one-to-one on D: for any two θ_1, θ_2, $(0,\theta_1)$ and $(0,\theta_2)$ are two different points in D, but they have the same image $(0,0)$ under T.

Practice with image and one-to-one

Let $T(r, \theta) = (r\cos\theta, r\sin\theta)$.
Let $D = (0, \infty) \times [0, 2\pi)$. Is the image of T all of $\mathbb{R}^2$? Is T one-to-one on D?

(a) Yes and yes.

(b) Yes and no.

(c) No and yes.

(d) No and no.

(e) I don't know.

$T(D) = \mathbb{R}^2 \setminus \{0\}$
T is one-to-one on D..

Practice with the Jacobian

Let $T(u, v) = (\frac{u^2}{v}, \frac{v}{u})$. Find $\frac{\partial(x,y)}{\partial(u,v)}$.

(a) $\frac{3}{v}$

(b) $\frac{1}{v}$

(c) $2v + \frac{1}{v}$

(d) $u + v$

(e) I don't know how.

Solution

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} x_u & x_v \\ y_u & u_v \end{vmatrix} \\
&= \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} \\
&= \frac{2}{v} - \frac{1}{v} \\
&= \frac{1}{v}.
\end{aligned}$$

Change of coordinates in three-dimensions

Theorem

Let T be a transformation from $D \subset \mathbb{R}^3$ to $\mathbb{R}^3$ such that

- *D and $T(D)$ are "nice";*
- *$\frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$ and T is one-to-one on D except possibly on the boundary.*

Suppose f is a continous function on $T(D)$. Then

$$\iiint_{T(D)} f(x,y,z)\,dV_{xyz} = \iiint_D f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV_{uvw}.$$