MATH 241 - LECTURE 27 - LINEAR CHANGE OF VARIABLES. §15.9

Friday, March 29

Last time - Spherical coordinates.

$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$

Let C be the region between $z = \sqrt{x^2+y^2}$ and $z = \sqrt{9-x^2-y^2}$

Sketch C, and write C in spherical coordinates.

Recall: For a "spherical wedge" $B = \left\{(\rho,\theta,\phi) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ a \le \rho \le b \\ c \le \phi \le d \end{array}\right\}$

and $f: B \to \mathbb{R}$ continuous,

$$\iiint_B f\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

on slide.

Example: Find the volume of C from above. (on slides)

TODAY: Linear change of variables (§15.10)

Recall: one-dimensional change of variables / substitution.

$$\int_a^b f(\underline{g(u)})\,\underline{g'(u)}\,du = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Three changes

- in 1d the advantage is to simplify the function
- in higher dimensions, we simplify the function & the region of integration

(cf polar, spherical, cylindrical coordinates)

Definition: $T: \underset{(u,v)}{\mathbb{R}^2} \longrightarrow \underset{(x,y)}{\mathbb{R}^2}$ is called linear

if it is of the form $T(u,v) = (\underbrace{au+bv}_{x}, \underbrace{cu+dv}_{y})$, $a,b,c,d \in \mathbb{R}$ constants.

Key properties: (1) $T(0,0) = (0,0)$

(2) T is determined by knowing

$T(1,0) = (a,c)$ & $T(0,1) = (b,d)$.

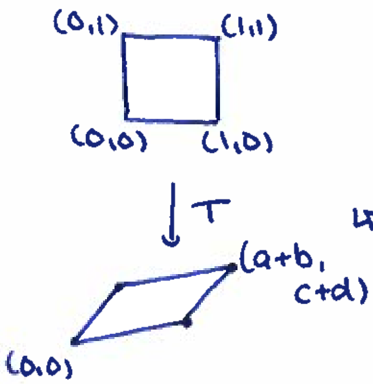
(3) T takes lines to lines.

(4) $T(\alpha \vec{u} + \vec{v}) = \alpha T(\vec{u}) + \vec{v}$ for $\alpha \in \mathbb{R}$.

Definition: if $D \subset \mathbb{R}^2$, the image of D under T is

$$T(D) = \{ T(u,v) \mid (u,v) \in D \}.$$

Examples: Let $S = [0,1] \times [0,1]$.

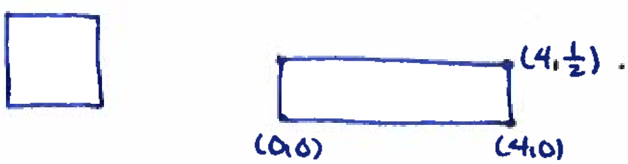


$T(1,1) = (a+b, c+d)$.
$= T(1,0) + T(0,1)$.

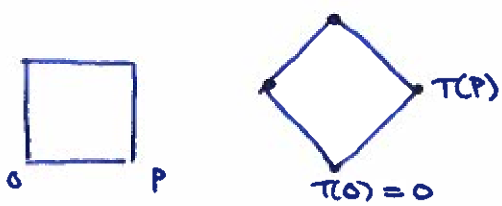
$\Rightarrow$ $T(S)$ is a parallelogram with ~~edges~~ vertices ~~$\langle a,c \rangle$ and $\langle b,d \rangle$~~

$(0,0)$, (a,c), (b,d) and $(a+c, b+d)$.

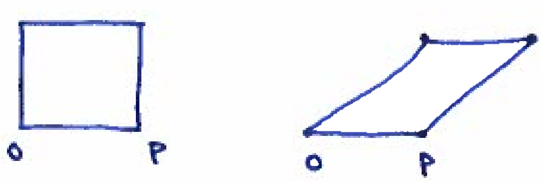
(1) $T(u,v) = (4u, \frac{1}{2}v)$ (stretch)



(2) $T(u,v) = \frac{1}{\sqrt{2}}(u+v, u-v)$. (rotate)



(3) $T(u,v) = (u+v, v)$. (shear)



Definition: The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

<u>Geometric meaning</u>: Area(S) = 1

Area$(T(S))$ = $|ad - bc| = \dfrac{\partial(x,y)}{\partial(u,v)}$.

$\Rightarrow$ T changes the area by $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

$\boxed{2}$ Q. Find $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$ for (1), (2), (3). Which is the smallest?

[see slides for solution]

<u>Theorem</u>: Given $D \subset \mathbb{R}^2$ and $f : T(D) \to \mathbb{R}$ continuous,

$$\iint_{T(D)} f(x,y)\, dA_{xy} = \iint_D f(T(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$ (see slides)

<u>Why?</u> We compute $\iint_{T(D)} f(x,y)\,dx\,dy$ by dividing D into small boxes $\square$ of area ΔA = Area$(\square)$.

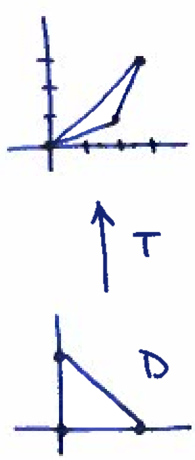
Then $T(D)$ is divided into small parallelograms $T(\square)$ of area $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| \Delta A$.

and we find $\iint_{T(D)} f\,dx\,dy$ by choosing points $(u^*,v^*) \in D$ giving points $T(u^*,v^*) \in T(D)$.

Then we sum the contributions

$$f(T(u^*,v^*)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta A$$

<u>Example</u>: B is the triangle with vertices $(0,0)$, $(2,1)$ & $(3,3)$.



Use $T(u,v) = (2u + 3v,\ u + 3v)$ to find $\iint_B (x-y)\,dA$.

• $T(0,0) = (0,0)$ $T(1,0) = (2,1) = (a,c)$ $T(0,1) = (3,3) = (b,d)$

$\Rightarrow B = T(D)$,

$D = \left\{(u,v) \,\middle|\, \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1-u \end{array}\right\}$.

• $f(x,y) = x - y$

$\Rightarrow f(T(u,v)) = f(2u+3v,\ u+3v) = (2u+3v) - (u+3v) = u$

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$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 6 - 3 = 3$$

$$\Rightarrow \iint_B f(x,y)\, \overset{dA}{\cancel{dx\,dy}} = \iint_D u \cdot 3\, \overset{dA}{\cancel{du\,dv}}\, dv\,du$$

$$= \int_0^1 \int_0^{1-u} 3u \, dv\, du = \int_0^1 3u[1-u]\, du = \int_0^1 3u - 3u^2\, du$$

$$= \cancel{\int} \left[\frac{3}{2}u^2 - u^3\right]_0^1 = \frac{1}{2}.$$