Last time: Polar rectangles.

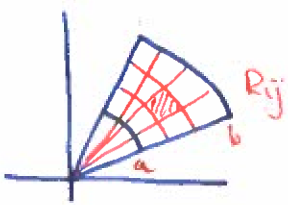
Let $R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$ be a polar rectangle.

Divide it into smaller polar rectangles

- divide $[a,b]$ into $[r_{i-1}, r_i]$ of length $\Delta r = \frac{b-a}{m}$
- divide $[\alpha,\beta]$ into $[\theta_{j-1}, \theta_j]$ of length $\Delta\theta = \frac{\beta-\alpha}{n}$

Let R_{ij} be the polar rectangle

$$\{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$



$\boxed{1}$ Use the fact that the area of a segment of a circle of radius r with central angle θ is $\frac{1}{2}r^2\theta$ to find ΔA_i = area of R_{ij}.

- See slide for description of integration.

Let's choose (x_{ij}^*, y_{ij}^*) to lie in the middle of R_{ij} as follows:

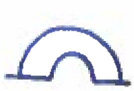
$$r_i^* = \frac{r_{i-1}+r_i}{2}, \qquad \theta_j^* = \frac{\theta_{j-1}+\theta_j}{2}$$

$$(x_{ij}^*, y_{ij}^*) = (r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*).$$

Note: $\Delta A_i = r_i^* \Delta r \Delta\theta$.

So
$$\boxed{\iint_D f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^* \,\Delta r\,\Delta\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Example: Let $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ 0 \le y\}$.



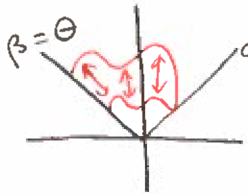
↳ in polar coordinates,

$$D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}.$$

What is $\iint_D y\,dA$?

[see slides for solution]

More general regions:



$$D = \{ (r,\theta) \mid \alpha \le \theta \le \beta, \quad h_1(\theta) \le r \le h_2(\theta) \}$$

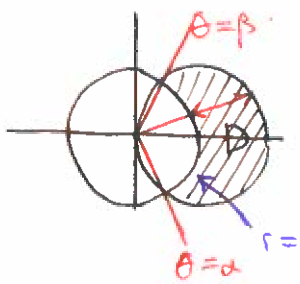
where $\beta - \alpha \le 2\pi$.

h_1, h_2 are continuous.

Theorem. Let f be a continuous function on D as above.

Then $\iint_D f \, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$.

Example: Set up the integral to find the area inside $(x-1)^2 + y^2 = 1$ and outside $x^2+y^2=1$.



$r=1$ lower bound.

- $r \ge 1$.
- what is the outer boundary?

$$(x-1)^2 + y^2 = 1$$

$$\Leftrightarrow x^2 - 2x + y^2 = 0$$

$$\Leftrightarrow r^2 - 2r\cos\theta = 0.$$

$$(r \ne 0) \quad \Leftrightarrow r = 2\cos\theta.$$

so the outer boundary gives the upper bound

$$r \le 2\cos\theta.$$

Intersection points: $2\cos\theta = 1$

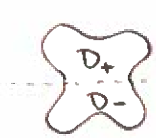
$$\Leftrightarrow \theta = \pm\frac{\pi}{3}.$$

So $D = \{ (r,\theta) \mid -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}, \quad 1 \le r \le 2\cos\theta \}$.

$$\Rightarrow \text{Area}(D) = \iint_D 1 \, dA = \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r \, dr \, d\theta.$$

Symmetry trick: $\iint_D f \, dA = 0$

if $f(x,-y) = -f(x,y)$ and $(x,y) \in D \Leftrightarrow (x,-y) \in D$.



$$\iint_{D_+} f \, dA = -\iint_{D_-} f \, dA.$$

(Similar arguments work for

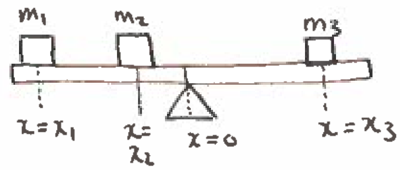
$$f(x,y) = -f(-x,-y)$$

or $f(x,y) = -f(-x,y)$.

Recall: even vs odd functions in CalcII.)

§ APPLICATIONS OF DOUBLE INTEGRALS

Centre of mass:



Balanced means $m_1x_1 + m_2x_2 + m_3x_3 = 0$.

Problem: Given some masses m_i at locations x_i, where is the balance point?

⇨ Find x_0 s.t.

$$m_1(x_1 - x_0) + m_2(x_2 - x_0) + m_3(x_3 - x_0) = 0.$$

$$\iff m_1x_1 + m_2x_2 + m_3x_3 = \underbrace{(m_1 + m_2 + m_3)}_{\text{total mass}} x_0$$

$$\Rightarrow x_0 = \frac{m_1x_1 + m_2x_2 + m_3x_3}{\text{total mass}}. \quad \leftarrow \textbf{centre of mass}.$$

Recall: For a wire of shape C with density function $\rho(x,y)$,

we had centre of mass $(\bar{x}, \bar{y})$

with $\bar{x} = \dfrac{\int_C \rho \cdot x \, ds}{\int_C \rho \, ds}$ $\quad \bar{y} = \dfrac{\int_C \rho \cdot y \, ds}{\int_C \rho \, ds}$.

Now we assume we have a flat object ("lamina") taking up a shape $D \subset \mathbb{R}^2$, with density $\rho(x,y)$ (in unit mass/unit area).

$$M_x = \frac{\iint_D x\rho(x,y)\, dA}{\iint_D \rho(x,y)\, dA} \leftarrow \textbf{total mass} \qquad M_y = \frac{\iint_D y\rho(x,y)\, dA}{\iint_D \rho(x,y)\, dA}.$$

centre of mass is (M_x, M_y)

Moment of inertia:

- about the x-axis $\quad I_x = \iint_D y^2 \rho(x,y)\, dA$

  y^2: square of distance from (x,y) to the x axis.

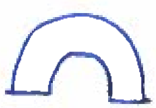
- about the y-axis $\quad I_y = \iint_D x^2 \rho(x,y)\, dA$

  x^2: " " from (x,y) to y axis

- about the origin $\quad I_0 = I_x + I_y = \iint_D (x^2+y^2)\, \rho(x,y)\, dA$.

  x^2+y^2: " " from (x,y) to origin.

Example: Consider a thin plate with constant density ρ_0 and shape $R = \{1 \le x^2+y^2 \le 4,\ 0 \le y\}$.



Estimate the centre of mass $\boxed{L}$

(See slides for solution)