Monday, 25 Feb. 2019

Last time: arc length $L = \int_a^b |\vec{r}\,'(t)|\,dt$.

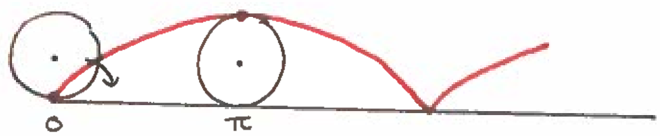
$\boxed{?}$

Example of a plane curve: ~~cycloid~~.
cycloid

- wheel of radius 1 rolls at 1 radian/second.

$\vec{r}(t)$ = position of an LED on the rim at time t.

(at $t=0$, centre of the wheel is at $(0,1)$ and LED is at $(0,0)$)



- At time t, centre is at $(t,1)$.

and the vector from the centre to the LED is $\langle -\sin t, -\cos t\rangle$.

$\Rightarrow \vec{r}(t) = \langle t,1\rangle + \langle -\sin t, -\cos t\rangle = \langle t-\sin t, 1-\cos t\rangle$

Upside-down, the cycloid has special properties:

- tautochrone
- brachistochrone (see slides - click for videos)

- this is an infinite-dimensional min/max problem (calculus of variations)

§16.2. INTEGRATING FUNCTIONS ALONG CURVES.

For today, assume curves are smooth

i.e. $\vec{r}\,'(t)$ is continuous and non-vanishing.

Integration with one variable. [see slides]

- Fix $g: [a,b] \to \mathbb{R}$

- divide $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of size $\Delta x = \frac{b-a}{n}$.
- for each i choose $x_i^* \in [x_{i-1}, x_i]$

$\leadsto \int_a^b g(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n g(x_i^*)\,\Delta x$

(if limit exists and doesn't depend on choices x_i^*)

Geometric meanings (see slides)

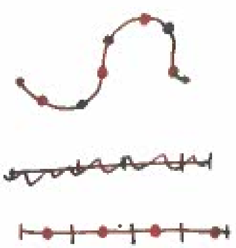
- $\int_a^b g(x)dx = (b-a)\cdot(\text{average value of } g \text{ on } [a,b])$
- $g \geq 0 \Rightarrow$ area under the curve.
- mass of a wire; can find its centre of mass.

Integration over curves:

Fix a curve $C \subset \mathbb{R}^2$ parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$.

Fix a function $f: C \to \mathbb{R}$.

- Divide $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of length $\frac{b-a}{n}$ and choose $t_i^* \in [t_{i-1}, t_i]$ as before.
- Let Δs_i be the length of C from $\vec{r}(t_{i-1})$ to $\vec{r}(t_i)$.



Definition
$$\int_C f\,ds = \lim_{n\to\infty} \sum_{i=1}^n f(\vec{r}(t_i^*))\,\Delta s_i$$

(assuming the limit exists and is independent of t_i^*)

Theorem:
$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$
$$= \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

WARNING - don't trace over the path more than once!

proof: $\Delta s_i \approx |\vec{r}(t_i) - \vec{r}(t_{i-1})| \approx |\vec{r}\,'(t_i^*)|\,\Delta t$.

$$\therefore \int_C f\,ds = \lim_{n\to\infty} \sum_{i=0}^n f(\vec{r}(t_i^*))\,\Delta s_i$$
$$= \lim_{n\to\infty} \sum_{i=0}^n f(\vec{r}(t_i^*))\,|\vec{r}\,'(t_i^*)|\,\Delta t$$
$$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt. \quad \square$$

Note: $\int_C f\,ds$ doesn't depend on the choice of parametrization $\vec{r}$.

(e.g. semi-circle from last time)

Geometric meanings:

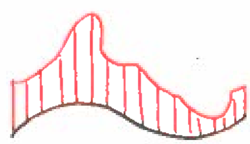
1) $\int_C f\,ds$ = (average value of f on C) · (length of C).

· in particular, if $f = 1$ everywhere

$\int_C ds$ = length of C

($= \int |\vec{r}\,'(t)|\,dt$ as we saw Friday)

2) if $f \geq 0$, f describes a fence of varying height over the curve C



C

$\int_C f\,ds$ = surface area of (one side of) the fence.

3) if f is the linear density of a wire shaped like C,

$\int_C f\,ds$ = total mass of wire.

Centre of mass of wire is at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\int_C x f\,ds}{\int_C f\,ds} \qquad \bar{y} = \frac{\int_C y f\,ds}{\int_C f\,ds}.$$

Example: Wire of constant density ρ over the semi-circle

$x^2 + y^2 = 1$, $y \geq 0$.

[?] Use geometric reasoning to guess the most likely option for the centre of mass.

Now let's calculate the centre of mass.

· $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$. $|\vec{r}\,'(t)| = 1$

$f(t) = \rho$ constant.

$$\Rightarrow \int_C f\,ds = \int_0^{\pi} \rho\,dt = \rho\pi.$$

· $\int_C x f\,ds = \int_0^{\pi} \cos t\,dt = [\sin t]_0^{\pi} = 0 \quad \Rightarrow \bar{x} = 0$

· $\int_C y f\,ds = \int_0^{\pi} \sin t\,dt = [-\cos t]_0^{\pi} = 2\pi \quad \Rightarrow \bar{y} = 2/\pi.$