Tangent planes, local max/min.

Last time: Recall the gradient of $f: \mathbb{R}^3 \to \mathbb{R}$:

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

$\boxed{?}$ Consider $f(x,y,z) = x^2 + y^2 + z^2$

Find the plane through $(1,1,2)$ perpendicular to $\nabla f(1,1,2)$.

Fact: Given $f: \mathbb{R}^n \to \mathbb{R}$, (differentiable), and

$\vec{a} = (a_1, \dots, a_n)$ a point in the level set $S_k = \{f(\vec{x}) = k\}$ of f.

Then the gradient $\nabla f(\vec{a})$ is perpendicular to S_k at $\vec{a}$.

Basic reason: Let $\vec{u}$ be any vector (with tail at $\vec{a}$) tangent to the level set S_k.

Since f is constant on S_k, $D_{\vec{u}} f(\vec{a}) = 0$.

But $D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$. so $\nabla f(\vec{a}) \perp \vec{u}$.

More carefully: $\cdot f: \mathbb{R}^2 \to \mathbb{R}$.

let $C = (x(t), y(t))$ be a parametrized curve in $\mathbb{R}^2$

- if $(x(t), y(t))$ is position at time t, $\vec{v} = \langle x'(t), y'(t)\rangle$ is the velocity.
- tangent line to C at $(x(t_0), y(t_0))$ is the line which
  - passes through $(x(t_0), y(t_0))$
  - is parallel to $\vec{v}_0 = \langle x'(t_0), y'(t_0)\rangle$

i.e. $$\begin{cases} x = x(t_0) + x'(t_0)(x - x_0) \\ y = y(t_0) + y'(t_0)(y - y_0) \end{cases}$$

Now consider $h(t) = f(x(t), y(t))$, and assume the whole curve C lies in the level set $\{f = k\}$.

$\Rightarrow h(t) = k \quad \forall t$, so $h'(t) = 0 \quad \forall t$.

But $h'(t) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$
$= \nabla f \cdot \vec{v}$.

$\Rightarrow$ $\nabla f(P)$ is perpendicular to $\vec{v}(P)$, and hence to the level curve $\{f = k\}$.

Likewise for $f: \mathbb{R}^3 \to \mathbb{R}$; let $S_k = \{f = k\}$ level surface.

· let $(x(t), y(t), z(t))$ be any parametrized curve in S_k

↳ tangent line at $(x(t_0), y(t_0), z(t_0))$ is

$$\begin{cases} x = x(t_0) + x'(t_0)(t - t_0) \\ y = y(t_0) + y'(t_0)(t - t_0) \\ z = z(t_0) + z'(t_0)(t - t_0). \end{cases} \qquad \vec{v} = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

The same argument shows

$$\nabla f(P) \cdot \vec{v} = 0.$$

this holds for any curve in S_k.

$\Rightarrow$ $\nabla f(P)$ is perpendicular to S_k.

§ TANGENT PLANES.

Remark: So far, we have defined the tangent plane to a surface which is the graph of a function $f: \mathbb{R}^2 \to \mathbb{R}$.

What about other surfaces?

e.g. the level surface S_k of a function $f: \mathbb{R}^3 \to \mathbb{R}$.

Motivation: for $f: \mathbb{R}^2 \to \mathbb{R}$, the tangent line to the curve $f = k$ at a point (x_0, y_0) is the line through (x_0, y_0) perpendicular to $\nabla f(x_0, y_0)$.

Definition: For $f: \mathbb{R}^3 \to \mathbb{R}$, the tangent plane to $f(x,y,z) = k$ at a point (x_0, y_0, z_0) is the plane which

- passes through (x_0, y_0, z_0)
- has normal vector $\vec{n} = \nabla f(x_0, y_0, z_0)$

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0.}$$

Rmk: the tangent plane contains the tangent line to any curve in $f(x,y,z) = k$.

$\boxed{2}$ Let S be a sphere with centre $O = (0,0,0)$. 12.3

Let $P \in S$. Claim: the tangent plane to S at P has normal vector $\vec{OP}$.

Example: Fix $f: \mathbb{R}^2 \to \mathbb{R}$ (so we know the tangent planes to $\Gamma(f)$ at points $(x,y,f(x,y))$.)

- Let $g(x,y,z) = f(x,y) - z$.

So the level set $S_0 = \{(x,y,z) \mid f(x,y) - z = 0\}$ is exactly the graph $\Gamma(f)$.

Does our new definition of tangent planes to S_0 agree with our old one?

Fix (x_0, y_0), let $z_0 = f(x_0, y_0)$, so $(x_0, y_0, z_0) \in S_0$.

$$\nabla g(x_0, y_0, z_0) = \left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1 \right\rangle$$

$\Rightarrow$ Tangent plane to S_0 at (x_0, y_0, z_0) is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

i.e. $z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$. :)

§ OPTIMIZATION. $\to$ finding max/min values for f. (not assumed differentiable).

Recall: for $f: \mathbb{R} \to \mathbb{R}$, a is a critical point if

- $f'(a) = 0$ OR $f(a)$ is not defined.

~~if f is a local maximum~~.

- if f achieves a local maximum ~~at~~ / minimum at $x = a$, then a is a critical point, but not all critical points correspond to local maxima/minima.

Second derivative test. if $f'(a) = 0$ and

- $f''(a) > 0 \Rightarrow$ local min. at a
- $f''(a) < 0 \Rightarrow$ local max at a
- $f''(a) = 0$?? e.g. $x^4, -x^4, x^3$

$f: \mathbb{R}^2 \to \mathbb{R}$. 12.4

Def. $(a,b) \in \mathbb{R}^2$ is a critical point if

· $\nabla f(a,b) = \langle 0,0 \rangle$ or $\nabla f(a,b)$ is not define.

Definitions: Local maximum/minimum.

Theorem: If f has a local maximum or minimum at (a,b) then (a,b) is a critical point.

(Intuition: $\nabla f(a,b)$ must point in the direction of max. increase.
$\Rightarrow \nabla f(a,b) = \langle 0,0 \rangle$.)

proof Assume f has a local max. at (a,b).

$\Rightarrow g(x) = f(x,b)$ has a local max at $x=a$

$\Rightarrow g'(a) = f_x(a,b) = 0$ or is not defined.

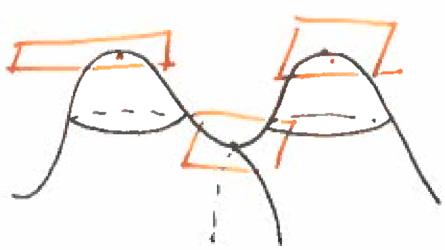
Likewise, $f_y(a,b)$ is 0 or undefined.

$\Rightarrow \nabla f(a,b)$ is $\langle 0,0 \rangle$ or undefined. ⊠.

Note: Tangent plane to f at (a,b) is
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

So it's horizontal $\Leftrightarrow \nabla f(a,b) = \langle 0,0 \rangle$.

Examples:



← local max

← neither local max/min