MATH 241 - LECTURE 9 § APPLICATIONS OF PARTIAL DERIVATIVES (14.3, 14.4)

Mon. 4 Feb, 2018

1 . Last time: Partial derivatives

E.g. Compute $f_x(1,2)$ where $f(x,y) = xe^{xy}$

2 look at the ~~g~~ contour graph of $f(x,t)$. [dark: negative light: positive] At the point $(x,t) = (\pi/2, 1.25)$ what can you say about the partial derivatives?

§ PARTIAL DIFFERENTIAL EQUATIONS (PDEs).

- In one variable we have "ordinary differential equations". (ODEs.)

Example: $P(t)$ = population at time t.

$P'(t) = cP(t)$

$\Rightarrow \quad P(t) = P_0 e^{ct}$

P_0: population at $t=0$.

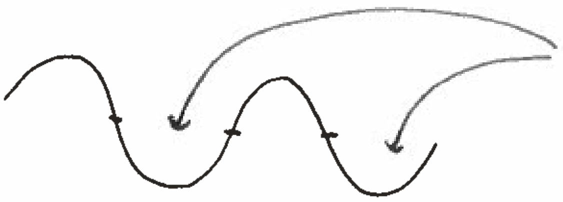
- in several variables we get ~~examples~~ equations involving partial derivatives.

e.g. if $u(x,t)$ is the temperature in a rod at position x and time t:

it turns out u satisfies the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad] \text{ Heat equation}$$

Say temperature at time $t=0$ is given by $u(x,0) = \sin x$



- where $\frac{\partial^2 u}{\partial x^2} > 0$, u increases with time
- where $\frac{\partial^2 u}{\partial x^2} < 0$ u decreases.

(One) Solution: $u(x,t) = e^{-t}\sin x$

Check: $u_t = -e^{-t}\sin x$

$u_x = e^{-t}\cos x$

$u_{xx} = -e^{-t}\sin x$ ☺

[Show image]

Rmk: In this course, we don't need to find solutions to PDEs; we just need to verify whether a given function is a solution or not.

§ LINEAR APPROXIMATIONS.

Definition: the linearization of $f: \mathbb{R}^2 \to \mathbb{R}$ at (a,b) is

$$\boxed{L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)}$$

Remarks: • L is a linear function

• $L(a,b) = f(a,b)$

• $\frac{\partial L}{\partial x}(a,b) = \frac{\partial f}{\partial x}(a,b)$; $\frac{\partial L}{\partial y}(a,b) = \frac{\partial f}{\partial y}(a,b)$.

$\Rightarrow$ L is the linear function that's most like f near (a,b).

$\hookrightarrow$ for (x,y) near (a,b) we can approximate f by L:

$$f(x,y) \approx L(x,y) \qquad \text{"linear approximation"}$$

Example: $f(x,y) = xe^{xy}$ $\to$ ~~linear approximation~~ linearization at $(1,0)$.

$f_x(x,y) = e^{xy} + xy\,e^{xy}$ $\Rightarrow f_x(1,0) = 1 + 0 = 1$.

$f_y(x,y) = x^2 e^{xy}$ $\Rightarrow f_{xy}(1,0) = 1$.

$\Rightarrow L(x,y) = 1 + 1(x-1) + 1(y-0)$

$= x + y$

Approximate $f(x,y)$ at $(1.1, -0.1)$:

$f(1.1, -0.1) \approx 1.1 + (-0.1) = 1$. $\boxed{2}$

(Actual answer: $f(1.1, -0.1) = 0.98542\ldots$)

The error in the linear approximation at $(a+\Delta x, b+\Delta y)$ is

$$E(\Delta x, \Delta y) := f(a+\Delta x, b+\Delta y) - L(a+\Delta x, b+\Delta y).$$

Definition: f is differentiable at (a,b)

$$\Leftrightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{E(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

9.4 , L is a good approximation to f near (a,b).

↪ if we zoom in on the graph of f, we get the graph of L

Theorem: If f_x and f_y exist near (a,b), and are continuous at (a,b), then f is differentiable at (a,b).

↳ may fail if f_x, f_y are not continuous.

Example ~~Example~~. [Skip in lecture if low on time]

$$f(x,y) = \begin{cases} xy/_{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

- $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0.$
- likewise $f_y(0,0) = 0$.

But f_x, f_y are not continuous:

for $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{y}{x^2+y^2} - \dfrac{xy}{(x^2+y^2)^2} 2x$

$$= \frac{y^3 - x^2 y}{(x^2+y^2)^2}$$

$\lim_{x\to 0} f_x(x,0) = 0$

$\lim_{y\to 0} f_x(0,y) = \lim_{y\to 0} \dfrac{y^3}{y^4}$ ~~=~~ $\dfrac{1}{y} \to \infty$.

The linear approximation would be $f(x,y) \approx 0$ near $(0,0)$.

But on the line $x=y$, $f(x,x) = \frac{1}{2}$.

So this is a bad approximation.

§ TANGENT PLANES.

Let $f: \mathbb{R}^2 \to \mathbb{R}$; let $c = f(a,b)$, so (a,b,c) is a point on the surface $\Gamma(f)$.

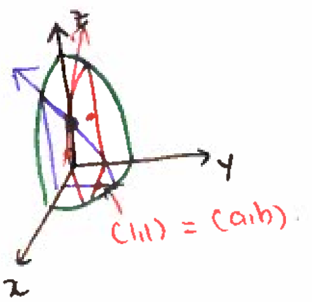
ne tangent plane to f at (a,b,c) is the graph of the linear approximation to f at (a,b,c).

$$\boxed{z = c + \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b).}$$

Example: $f(x,y) = 4 - x^2 - 2y^2$ — paraboloid

$\frac{\partial f}{\partial x}(x,y) = -2x \quad \Rightarrow \quad \frac{\partial f}{\partial x}(1,1) = -2$

$\frac{\partial f}{\partial y}(x,y) = -4y \quad \Rightarrow \quad \frac{\partial f}{\partial x}(1,1) = -4.$



$\therefore\ c = 4 - 1 - 2 = 1.$

$\Rightarrow$ tangent plane is $z = 1 - 2(x-1) - 4(y-1)$

i.e. $2x + 4y + z = 7$

- Slice at $y = b = 1$

  $\hookrightarrow f(x,1) = 2 - x^2$ parabola.

  tangent line at $(1,1)$ has slope $\frac{\partial f}{\partial x}(1,1)$

- Slice at $x = a = 1$

  $\hookrightarrow f(1,y) = 3 - 2y^2$ parabola

  tangent line at $(1,1)$ has slope $\frac{\partial f}{\partial y}(1,1)$

tangent plane is the unique plane containing both these ~~vectors~~ lines

and in fact all tangent lines to any curve in $\Gamma(f)$ passing through $(a,b) = (1,1)$.