<u>Objectives:</u> • Define the limit of a function of several variables.

• Find the limit, or show that it doesn't exist

• look at values along curves

• use the Squeeze Theorem

• use Polar Coordinates.

<u>Recall from Monday:</u>

• Given $E:\mathbb{R}\to\mathbb{R}$, we say the "limit of E as h approaches 0 is 0" if:

• for any $\varepsilon>0$
we can find some $\delta>0$ (depending on ε)
such that if $h\in\mathbb{R}$ satisfies $0<|h|<\delta$,
then $E(h)$ satisfies $|E(h)|<\varepsilon$.

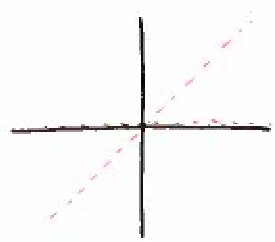
<u>Notation</u> $\lim_{h\to 0} E(h) = 0$.

More generally, given $f:\mathbb{R}\to\mathbb{R}$, we say $\lim_{x\to a} f(x) = L$ if
we can write $f(a+h) = L + E(h)$, where $E:\mathbb{R}\to\mathbb{R}$ is a function with $\lim_{h\to 0} E(h) = 0$.

• Equivalently:
• for any $\varepsilon>0$, we can find some $\delta>0$ such that if $h\in\mathbb{R}$ satisfies $0<|h|<\delta$, then $f(a+h)$ satisfies $|f(a+h)-L|<\varepsilon$.

(i.e. $a+h$ "close to a" $\Rightarrow$ $f(a+h)$ "close to L")

<u>Example:</u> $f(x) = \begin{cases} 0, & x\in\mathbb{Q} \\ x, & x\notin\mathbb{Q}\end{cases}$



1) $\lim_{x\to 0} f(x) = 0$:

• Given $\varepsilon = \frac{1}{10}$ say, we can take $\delta = \frac{1}{10}$:

↳ if ~~the~~ $0 < |h| < \frac{1}{10}$, then

$f(h)$ is 0 or h, which are both of absolute value $< \frac{1}{10}$.

$\uparrow$

$f(a+h)$

$\uparrow$

$a=0$ in this example

$\Rightarrow \quad |f(h) - 0| = |f(h)| < \frac{1}{10} = \varepsilon.$

$\uparrow$

$L=0$ in this example

What about a general $\varepsilon > 0$.

The same argument ~~show~~ that we can again take $\delta = \varepsilon$.

$\therefore \lim_{x\to 0} f(x) = 0$ as claimed.

(2) $\lim_{x\to 1} f(x)$ does not exist:

- indeed given $\varepsilon = \frac{1}{2}$, for example:

we cannot find any δ which is suitable.

No matter how small δ is, there is always a rational number h with $0 < \underset{|h|}{\underset{\shortparallel}{h}} < \delta$,

and $f(\underbrace{1+h}_{\in \mathbb{Q}}) = 0$

so $|f(1+h) - 1| = |1| \geq 1 > \varepsilon$.

Note: to prove that a limit doesn't exist, we only need to find ONE value of ε for which there is NO suitable choice of δ

But to prove that the limit does exist, we need to show that EVERY $\varepsilon > 0$ can be made to work

§. GENERALIZE TO MORE VARIABLES.

<u>Definition:</u> $f: \mathbb{R}^2 \to \mathbb{R}$ a function, $\vec{a} = \langle a_1, a_2 \rangle$ corresponding to (a_1, a_2) a point in $\mathbb{R}^2$

$L \in \mathbb{R}$ a number.

We say f has limit L as $\vec{x} = \langle x_1, x_2 \rangle$ approaches $\vec{a}$, and write

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

or

$$\lim_{(x_1, x_2) \to (a_1, a_2)} f(x_1, x_2) = L$$

if : for any $\varepsilon > 0$, we can make $f(x_1, x_2)$ ε-close to L:

more precisely, ~~there~~ is some $\delta > 0$ (depending on ε) such that

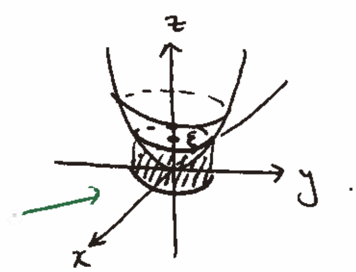
if (x_1, x_2) is δ-close to (a_1, a_2)

$$\left(\text{i.e. } \underbrace{|\langle x_1 - a_1, x_2 - a_2 \rangle|}_{= |\vec{x} - \vec{a}|} = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta\right)$$

$$0 < |\vec{x} - \vec{a}|$$

then $|f(x_1, x_2) - L| < \varepsilon$.

<u>Example 1:</u> $f(x, y) = x^2 + y^2$.



- if (x, y) is inside the shaded disk, then $0 \leq f(x,y) < \varepsilon$.
- δ = radius of the disk $= \sqrt{\varepsilon}$.

↳ So limit $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

<u>Example 2:</u> (this is in the video lecture).

$f(x, y) = x + 3y$.

<u>Claim:</u> $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

Given $\varepsilon > 0$, we want to find δ s.t. if $0 < |\vec{x}| < \delta$, then

$|f(x,y)| < \varepsilon$.

Strategy: try to bound $|f(\vec{x})|$ in terms of something involving $|\vec{x}|$:

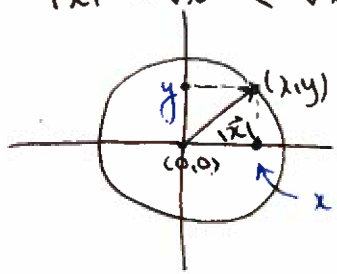
$$|f(\vec{x})| = |x + 3y| \leq |x| + 3|y|$$

triangle inequality

Note. $|x|, |y| \leq \sqrt{x^2+y^2} = |\vec{x}|$

Algebraic reason: • $|x| = \sqrt{x^2} \leq \sqrt{x^2 + \underbrace{y^2}_{\geq 0}}$

Geometric reason:



← circle of radius $|\vec{x}|$

Consequence: $|f(\vec{x})| \leq |x| + 3|y| \leq |\vec{x}| + 3|\vec{x}| = 4|\vec{x}|$.

So if we knew $|\vec{x}| < \frac{1}{4}\varepsilon$, we'd know

$$|f(\vec{x})| \leq 4 \cdot \frac{1}{4}\varepsilon = \varepsilon$$

i.e. we should take $\delta = \frac{1}{4}\varepsilon$.

Example 3: $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. [This example is also in the video]

Claim: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = |\vec{x}|.$$

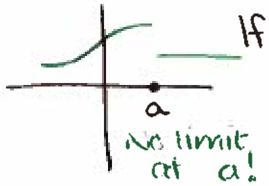
↳ Given $\varepsilon > 0$, we should take $\delta = \varepsilon$.

The above shows that if $0 < |\vec{x}| < \delta = \varepsilon$, then

$$|f(x,y)| \leq |\vec{x}| < \varepsilon.$$

So the limit is 0.

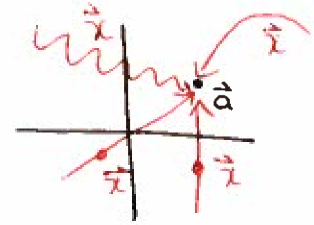
WARNING • For functions of 1 variable, we could look at limits as x approached a from the left and the right.



If both these limits exist and equal L, we ~~say~~ know that $\lim_{x \to a} f(x)$ exists and is equal to L.

But in 2-variables, $\vec{x}$ doesn't have to approach $\vec{a}$ from the left or the right.

There are infinitely many different paths along ~~each~~ which $\vec{x}$ can approach $\vec{a}$:



Example 4 [This example is in the video]

4. $f(x,y) = \dfrac{2xy}{x^2+y^2}$. limit as $(x,y) \to (0,0)$??

↖ not defined when $(x,y) = (0,0)$.

- Let's let (x,y) approach $(0,0)$ along the x-axis.

i.e. $y=0$, $(x,0) \to (0,0)$.

$$f(x,0) = \frac{2x \cdot 0}{x^2+0^2} = \frac{0}{x^2} = 0. \quad \text{constant}$$

$$\Rightarrow \lim_{\substack{\cancel{(x,0)\to(0,0)} \\ x \to 0}} f(x,0) = 0.$$

- Now let's approach $(0,0)$ along the line $y=x$:

$$f(x,x) = \frac{2x^2}{x^2+x^2} = 1.$$

$$\Rightarrow \lim_{x \to 0} f(x,x) = 0.$$

$\therefore$ the limit cannot exist.

<u>Theorem:</u> Suppose that $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a curve C_1

and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a curve C_2

<u>and</u> $L_1 \neq L_2$.

Then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 5. $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ $\qquad \lim_{(x,y)\to(0,0)} f(x,y) = ??$

Note: All straight lines through the origin have the form

$y = mx \quad (m \in \mathbb{R}) \quad$ or $\quad x = 0$.

- Along the line $x=0$

$$f(0,y) = \frac{0y^2}{0^2+y^4} = 0 \quad \xrightarrow[x\to 0]{} 0$$

- Along the line $y=mx$

$$f(x,mx) = \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2}$$

$$\xrightarrow[x\to 0]{} 0.$$

But this is not enough! (x,y) can travel along curved paths, not just straight ones.

e.g. let (x,y) approach $(0,0)$ along the parabola $x=y^2$:

$$f(y^2,y) = \frac{y^2y^2}{(y^2)^2+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}. \quad \xrightarrow[y\to 0.]{} \frac{1}{2}$$

So the limit doesn't exist.

[This is hard to visualize, but you don't need to be able to visualize it to prove that the limit doesn't exist

You can see this surface in the video]

The method of checking limits along curves allows us to prove that a limit doesn't exist.

It could also allow us to guess what we think the limit is if it does exist.

But we can never use this method to prove that the limit actually exists.

TECHNIQUES FOR PROVING LIMITS DO EXIST (and finding them)

1) Use what we know in 1d:

$$\lim_{(x,y)\to(a,b)} x = a. \qquad \lim_{(x,y)\to(a,b)} y = b.$$

2) <u>Limit laws</u>: if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L_1$

$\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = L_2$,

then (A) $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y))$ exists and is equal to $L_1 + L_2$.

(B) $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y)\,g(x,y))$ exists and is equal to $L_1 L_2$.

3) <u>SQUEEZE THEOREM</u>:

If we have $g, h : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is a disk around (a,b)

such that

$$g(x,y) \leq f(x,y) \leq h(x,y) \text{ for all } (x,y) \in D$$

and $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$

then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to L.

<u>Example 6</u> $f(x,y) = xy/\sqrt{x^2+y^2}$ (this is example 3 again, but a different method)

$|x| \leq \sqrt{x^2+y^2}$ as before

$$\Rightarrow \quad |f(x,y)| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}|y| = |y|$$

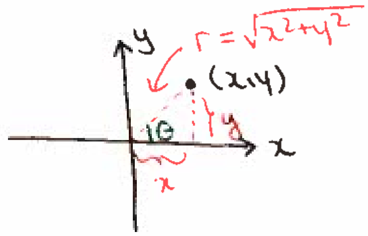
$$\Rightarrow \quad \underset{\uparrow\, g(x,y)}{-y} \leq f(x,y) \leq \underset{\uparrow\, h(x,y)}{y}$$

(Here $D = \mathbb{R}^2$)

Since $-y \xrightarrow[(x,y)\to 0]{} 0$, $y \xrightarrow[(x,y)\to 0]{} 0$, the Squeeze theorem tells us

$$f(x,y) \xrightarrow[(x,y)\to 0]{} 0.$$

§ ONE MORE TECHNIQUE - POLAR COORDINATES:



From fig. we have:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$(x,y) = (r\cos\theta, r\sin\theta)$$

↳ (x,y) are the Cartesian coordinates

(r,θ) are the polar coordinates

$r \geq 0$

$\theta \in [0, 2\pi)$

- $(x,y) \to 0 \iff r \to 0.$

Rewriting a function in terms of polar coordinates may make the behaviour around $(0,0)$ more obvious.

<u>Example 7</u> $f(x,y) = xy/\sqrt{x^2+y^2}$ (again!)

In polar coordinates:

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\, r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = r\cos\theta\sin\theta = \frac{r}{2}\sin 2\theta$$

$$\xrightarrow[r\to 0]{} 0$$

<u>Example 8</u>: $f(x,y) = \dfrac{x^2y^2}{(x^2+y^2)^2}$

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{(r^2\cos^2\theta + r^2\sin^2\theta)^2} = \frac{r^4\left(\frac{1}{2}\sin 2\theta\right)^2}{r^4} = \frac{1}{4}\sin^2(2\theta).$$

↳ this does not approach 0 as $r \to 0$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.