Math 241 - Lecture 6 - Monday, 28 Jan. 2019.
(§14.1, §12.6, §14.2)

1. Sketch the graph of the surface $f(x,y) = y^2$ in $\mathbb{R}^3$.
   - Draw the level sets in the xy plane for a few choices of $z=k$.
   - Draw the contour map
   - Sketch the ~~curve.~~ surface.

Definition: · Let $\mathbb{P}$ be a plane with normal vector $\vec{n}$, and let C be a curve in the plane ~~on~~ $\mathbb{P}$.
Then a cylinder on C is a surface consisting of all lines through C parallel to $\vec{n}$.
↑ "rulings"

Note: You can easily identify the formula for a cylinder whose rulings are parallel to a coordinate axis, because that variable will be missing from the formula.

§. BACK TO LEVEL SETS OF FUNCTIONS IN 3 VARIABLES.

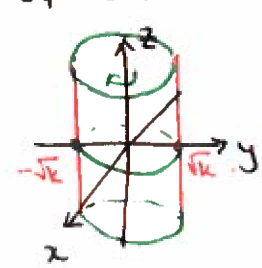
Example: level sets of $f(x,y,z) = x^2+y^2-z^2$.
↳ level sets $k = x^2+y^2-z^2$ will be surfaces, so we practice drawing surfaces again.

Helpful trick: this function is "symmetric around z"
i.e. the value of f doesn't really depend on (x,y), only the distance $r=\sqrt{x^2+y^2}$ of the point (x,y,z) from the z-axis.

$$f(x,y,z) = r^2 - z^2.$$

Simpler example: $g(x,y,z) = x^2+y^2$.
· level set $k = x^2+y^2$ is a circular cylinder

to graph it: 1) find values at $x=0$
2) rotate around z axis.



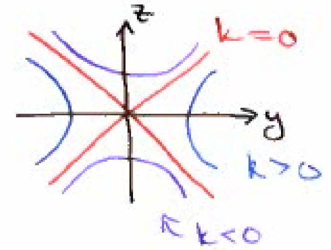
16.2

· Back to $f(x,y,z) = x^2+y^2-z^2$

Level set $x^2+y^2-z^2=k$

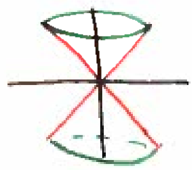
1 a) look at $x=0$: $y^2-z^2=k$.

Recall: contour map of this function



2) Rotate around z-axis:

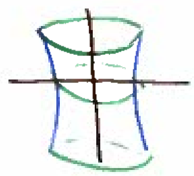
$k=0$



$x^2+y^2-z^2=0$

CONE.

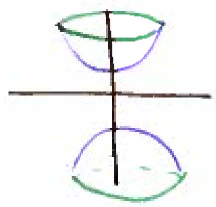
$k>0$



$x^2+y^2-z^2=k>0$

HYPERBOLOID OF 1 SHEET.

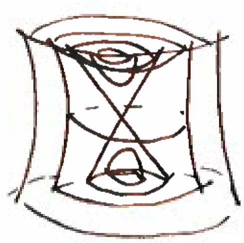
$k<0$



$x^2+y^2-z^2=k<0$

HYPERBOLOID OF 2 SHEETS.

So the contour map of $f(x,y,z)=x^2+y^2-z^2$ looks like



$\boxed{Q}$ Question on symmetry.

§ QUADRIC SURFACES.

Definition A quadric surface is given by the solutions to an equation which is a degree 2 polynomial equation in three variables.

General form: $Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$

Classification: (see worksheet)

- cylinder (C a conic)
- ellipsoid
- elliptic paraboloid
- hyperbolic paraboloid
- hyperboloid of 1 or 2 sheets
- cone.

$\boxed{1}$ Consider the equation $x^2 + 6x + 2y^2 + 2z^2 + 7 = 0$ 16.3

i.e. $\frac{(x+3)^2}{2} + y^2 + z^2 = 1$.

· Sketch level sets $z = k$

$\boxed{1}$ Consider ~~the~~ traces $x = k$, $y = k$. Sketch.

Remark: this function is symmetric about the x-axis.

§14.2 INTRO TO LIMITS.

(Start with 1-variable setting)

$E : \mathbb{R} \to \mathbb{R}$
$\uparrow$
"Error"

Def. We say that the limit of E (as h goes to 0) is 0 if

for every (challenge) $\varepsilon > 0$

there exists a (response) $\delta > 0$

such that any $h \in \mathbb{R}$ with $0 < |h| < \delta$ satisfies

$|E(h)| < \varepsilon$.

i.e. if our tolerance for error is ε, there is an input setting δ such that if our input meets the requirement δ, the error is less than ε.

Notation: $\lim_{h \to 0} E(h) = 0$.

Example: We're building a machine, and we need a $1\text{cm} \times 1\text{cm} \times 1\text{cm}$ cube of ~~metal~~ with density $1\text{g}/\text{cm}^3$.

If the cube weighs much ~~more than~~ or less than 1g then the machine will be off balance.

The factory sends us a cube of side length $1+h$.

$\Rightarrow \text{mass}(h) = (1+h)^3 \text{ g} = (1 + 3h + 3h^2 + h^3) \text{ g}$.

Error $E(h) = \text{mass}(h) - 1\text{g} = 3h + 3h^2 + h^3 \text{ g}$.

So if our tolerance for error is ε, we need to ensure we can find h s.t.

$|E(h)| < \varepsilon$.

e.g. $\varepsilon = \frac{1}{10}$ $\qquad h(3 + \underbrace{3h + h^2}) < \frac{1}{40}\cdot 4 = \frac{1}{10}.$

idea: make this < 4 and this less than $\frac{1}{40}$

note: if $h < \frac{1}{40}$, $3 + 3h + h^2 < 4$.

So we can take $\delta = \frac{1}{40}$.

or $\delta = \frac{1}{50}$ ...

But $\delta = \frac{1}{20}$ won't work.

e.g. $h = \frac{1}{30}$ has $0 < |h| < \delta$, but

$$\frac{1}{30}\left(3 + 3\cdot\frac{1}{30} + \left(\frac{1}{30}\right)^2\right) > \frac{1}{30}\cdot 3 = \frac{1}{10}.$$

More general strategy: given $\varepsilon > 0$, take $\delta = \min\left(\frac{1}{4}\varepsilon, \frac{1}{12}\right)$

$$0 < |h| < \delta \Rightarrow |E(h)| = |h(3 + 3h + h^2)|$$

$$\leq |h|\,(3 + |3h| + |h|^2)$$

$$< \frac{1}{4}\varepsilon\left(3 + \frac{1}{4} + \frac{1}{144}\right) < \frac{1}{4}\varepsilon\cdot 4 = \varepsilon.$$

So $\lim_{h\to 0} E(h) = 0$.

Example: $E(h) = \begin{cases} 0 & h \notin \mathbb{Q} \\ h & h \in \mathbb{Q} \end{cases}$

Given $\varepsilon > 0$. what can we choose for δ? $\boxed{L}$

Warning: $E(h) = \begin{cases} h & h \neq 0 \\ 10 & h = 0 \end{cases}$ has limit 0 as $h \to 0$

even though $E(h) = 10$.

Last example: $E(h) = \sin h$

$\hookrightarrow$ no limit at $h = 0$.

Definition We say $\lim_{x\to a} f(x) = c$ if $\exists\ E : \mathbb{R} \to \mathbb{R}$ with $\lim_{h\to 0} E(h) = 0$

and $f(a+h) = c + E(h)$.