MATH 595 Thursday 29 March

Riemann–Roch

(1) **Exercise IV.1.1 and IV.1.2**

Let X be a non-singular projective curve. Let $P = P_1, P_2, \ldots, P_r$ be points of X.

(a) Prove that there exists a non-constant rational function $f \in K(X)$ which is regular except at P.
(Hint: Think about the divisor (f) of such a function f. How can you use Riemann–Roch to help you?)

(b) Show that there exists a rational function f which has poles of some order at each P_i and which is rational everywhere else.

(2) **Exercise IV.1.5**

Let X be a curve of genus g, and let D be an effective divisor. Prove that $\dim|D| \leq \deg D$, with equality if and only if $D = 0$ or $g = 0$.

(Hint: use the short exact sequence for the subscheme $D \subset X$, maybe tensored by a suitable line bundle, to compare $\ell(K - D)$ with $\ell(K) = g$. Now use Riemann–Roch.)

(3) **Exercise IV.1.6, IV.1.7 (a)**

(a) Let X be a curve of genus g. Prove that there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g + 1$.
(Hint: note that $\deg f = \deg(f)_\infty$. Now fix P a point of X and consider the linear system $|(g + 1)P|$.)

(b) Now assume that X is an elliptic curve (i.e. $g = 1$). Prove that there is a finite degree two map $f : X \to \mathbb{P}^1$.

(c) Motivated by this, we call a curve X with $g \geq 2$ *hyperelliptic* if it has a finite morphism $f : X \to \mathbb{P}^1$ of degree 2. Prove that every curve of genus 2 is hyperelliptic, by showing that $|K|$ is a complete linear system of degree 2 and dimension 1 without basepoints.
(Hint: Recall that $P \in X$ is a basepoint of the linear system $|D|$ if for every $D' \in |D|$, D' is supported at P. Convince yourself that this means that $|D|$ and $|D - P|$ are complete linear systems of equal dimension.)