HW12 Solutions.

Exercise 1: a) Prove that the composition of Möbius transformations is again a Möbius transformation.

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

$$\Rightarrow \quad f_1 \circ f_2(z) = f_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) = \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + \cancel{b_2} d_1} \cdot \frac{c_2 z + d_2}{c_2 z + d_2}$$

$$= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + \overset{d_1}{\cancel{b_2}}(c_2 z + d_2)}$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)} \in \mathcal{M}.$$

(we know that $AD - BC \neq 0$ because we know $f_1 \circ f_2$ is invertible)

*can also show by hand that
$AD - BC = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0$

b) Rescale so $ad - bc = 1$.

Have $f(z) = \dfrac{az - b}{cz - d}$ with $ad - bc = D \neq 0$

let $\lambda = 1/\sqrt{D} \in \mathbb{C} \setminus \{0\}$. so $(\lambda a)(\lambda d) - (\lambda b)(\lambda c) = \lambda^2 (ad - bc)$

$$= \frac{1}{D} D = 1.$$

So we write $f = \dfrac{(\lambda a) z - \lambda b}{(\lambda c) z - \lambda d}$.

c) Compare to matrices. Give $\Phi: SL_2(\mathbb{C}) \to \mathcal{M}$ surjective.

Note: $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$

and the entries in the new matrix give the constants from (a).

Define $\Phi: SL_2(\mathbb{C}) \to \mathcal{M}$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto f(z) = \frac{az + b}{cz + d}.$$

By note, $\Phi(A \cdot B) = \Phi(A) \circ \Phi(B)$.

By (b) for any $f \in \mathcal{M}$, we can find $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ s.t. $ad - bc = 1$, so $A \in SL_2(\mathbb{C})$ and $f(z) = \Phi(A)$.

Exercise 2: $f(z) = \beta \dfrac{z-\alpha}{\bar{\alpha}z-1}$, $|\alpha|<1$, $|\beta|=1$.

(a) Show that if $|z|^2 = 1$, $|f(z)|^2 = 1$.

$|z|^2 = z\bar{z} = 1$.

$$|f(z)| = \left(\beta \frac{z-\alpha}{\bar{\alpha}z-1}\right)\cdot\left(\bar{\beta}\frac{\bar{z}-\bar{\alpha}}{\alpha\bar{z}-1}\right) = \underbrace{\beta\bar{\beta}}_{=1}\ \frac{\overbrace{z\bar{z}}^{=1} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha}}{\bar{\alpha}\alpha \underbrace{z\bar{z}}_{=1} - \bar{\alpha}z - \alpha\bar{z} + 1}$$

$$= \frac{-z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha}}{\alpha\bar{\alpha} - z\bar{\alpha} - \alpha\bar{z}} = 1$$

(b) Find $z \in \mathrm{Disk}_p$ (i.e. $|z|^2<1$) with $|f(z)|^2<1$.

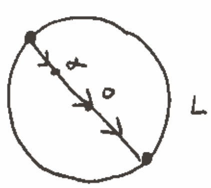
By assumption $|z|^2<1$ so we can take $z=\alpha$.

$$f(\alpha) = \beta\frac{\alpha-\alpha}{\bar{\alpha}\alpha-1} = 0.$$

$$|f(\alpha)|^2 = 0 < 1.$$

Exercise 3: $f(z) = \dfrac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha|<1$.

(a) Draw $L = \{t\alpha\}$



omega-points should be fixed.

$f(L) \subset L$.

(b) Prove that the omega points of L are fixed points of f.

want $t\alpha \in L$ with $|t\alpha| = 1$.

so we take $t = \pm\dfrac{\alpha}{|\alpha|}$

$$f\left(\frac{\alpha}{|\alpha|}\right) = \cancel{\frac{\left(\frac{\alpha}{|\alpha|}-\alpha\right)}{\left(1-\bar{\alpha}\frac{\alpha}{|\alpha|}\right)}} = \cancel{\frac{\alpha\left(\frac{1}{|\alpha|}-1\right)}{\frac{(|\alpha|-\bar{\alpha}\alpha)}{|\alpha|}}} = \cancel{\alpha\left(\frac{1}{|\alpha|}-1\right)}$$

$$= \frac{\left(\frac{\alpha}{|\alpha|}-\alpha\right)}{\left(1-\bar{\alpha}\frac{\alpha}{|\alpha|}\right)} = \frac{\frac{\alpha}{|\alpha|}(1-|\alpha|)}{1-|\alpha|} = \frac{\alpha}{|\alpha|} \quad \text{fixed!}$$

Likewise, $f\left(-\frac{\alpha}{|\alpha|}\right) = \dfrac{-\frac{\alpha}{|\alpha|}-\alpha}{1+\bar{\alpha}\frac{\alpha}{|\alpha|}} = \dfrac{-\frac{\alpha}{|\alpha|}(1+|\alpha|)}{1+|\alpha|} = -\frac{\alpha}{|\alpha|}$.

Also fixed!

(c) Prove that $f(L) \subset L$.

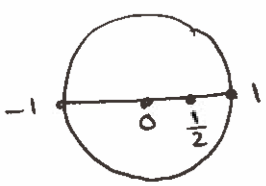
take $t\alpha \in L$.

$$f(t\alpha) = \frac{t\alpha-\alpha}{1-\bar{\alpha}t\alpha} = \alpha\underbrace{\left(\frac{t-1}{1-|\alpha|^2 t}\right)}_{=t'\in\mathbb{R}} \in L.$$

(d) Assume $\alpha = \frac{1}{2}$.

Prove $d_p(0,\alpha) = \ln 3$.

Prove $d_p(t\alpha, f(t\alpha)) = \ln 3 \quad \forall\, t \in \mathbb{R}$.



- $d_p(0, \frac{1}{2}) = |\ln(0, \frac{1}{2}, 1, -1)|$

$$= \left|\ln \frac{-1}{1}\cdot\frac{1/2-(-1)}{1/2-1}\right|$$

$$= \left|\ln - \frac{3/2}{-1/2}\right| = |\ln 3| = |\ln 3|$$

- $d_p(t\alpha, f(t\alpha)) = \left|\ln\left(\dfrac{t\alpha-1}{t\alpha+1}\quad\dfrac{f(t\alpha)+1}{f(t\alpha)-1}\right)\right|$

- $t\alpha = t/2$, $f(t\alpha) = \dfrac{(t\alpha)-\alpha}{1-\bar{\alpha}(t\alpha)} = \dfrac{t/2 - 1/2}{1-1/2 \cdot t/2} = \dfrac{2t-2}{4-t}$.

So $d_p(t\alpha, f(t\alpha)) = \left|\ln \dfrac{t/2-1}{t/2+1}\cdot\dfrac{\frac{2t-2}{4-t}+1}{\frac{2t-2}{4-t}-1}\right|$

$$= \left|\ln\frac{t-2}{t+2}\cdot\frac{2t-2+(4-t)}{2t-2-(4-t)}\right| = \left|\ln\frac{t-2}{t+2}\cdot\frac{t+2}{3t-6}\right|$$

$$= \left|\ln\frac{1}{3}\right| = |-\ln 3| = \ln 3.$$