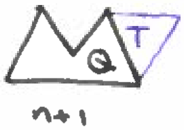
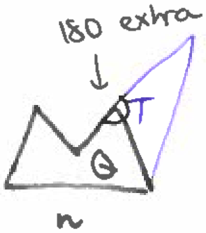
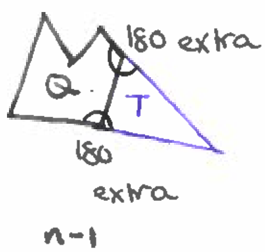
Exercise 1: A few more examples. SOLUTIONS TO HW 11.

- 

$$\begin{aligned}\text{defect}(P) &= 180(n-1) - (\text{angle sum } P)\\ &= 180(n-1) - (\text{angle sum } Q + \text{angle sum } T)\\ &= \big(180(n-2) - \text{angle sum of } Q\big)\\ &\quad + (180 - \text{angle sum of } T)\\ &= \text{defect}(Q) + \text{defect}(T)\end{aligned}$$

- 

$$\begin{aligned}\text{defect}(P) &= 180(n-2) - (\text{angle sum } P)\\ &= 180(n-2) - (\text{angle sum } Q + \text{angle sum } T - 180)\\ &= (180(n-2) - (\text{angle sum } Q))\\ &\quad + (180 - (\text{angle sum } T))\\ &= \text{defect}(Q) + \text{defect}(T).\end{aligned}$$

- 

$$\begin{aligned}\text{defect}(P) &= 180(n-3) - \text{angle sum of } P\\ &= 180(n-3) - (\text{angle sum of } Q + \text{angle sum of } T\\ &\qquad - 2(180))\\ &= \big(180(n-2) - \text{angle sum of } Q\big)\\ &\quad + (180 - \text{angle sum of } T)\\ &= \text{defect}(Q) + \text{defect}(T).\end{aligned}$$

- Induction argument:
- $n=1$ Δ nothing to check
- now suppose that we know that any polygon Q which is completely triangulated into n pieces $T_1, \dots, T_n$ has defect $\sum_{i=1}^{n} \text{defect } T_i$ (for fixed $n \geq 1$)
- and suppose P is triangulated into $T_1, \dots, T_{n+1}$.
  We know that at least one of the triangles, say T_{n+1}, has one or two free edges; call the union of the remaining triangles Q.
  Then (Q, T_{n+1}) are as in one of our six examples, so
  $\text{defect}(P) = \text{defect}(T_{n+1}) + \text{defect}(Q)$
  and by induction $\text{defect}(Q) = \sum_{i=1}^{n} \text{defect}(T_i)$. $\square$

Practice with ~~multiplying~~ complex numbers: (Exercise 2)

(a) $e^{i\phi} e^{i\psi} = (\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$

$= (\cos\phi\cos\psi - \sin\phi\sin\psi) + i(\sin\phi\cos\psi + \cos\phi\sin\psi)$

$= \cos(\phi+\psi) + i\sin(\phi+\psi)$

$= e^{i(\phi+\psi)}$

(Note: this uses trig identities in its proof, but from now on you can use this fact to remember the trig formulas)

(b) • $\frac{1}{2i} = \frac{1}{2i}\left(\frac{-2i}{-2i}\right) = \frac{-2i}{4} = -\frac{1}{2}i$

• $\frac{1+i}{1-i} = \frac{1+i}{1-i}\cdot\frac{1+i}{1+i} = \frac{1+2i-1}{1+1} = \frac{2i}{2} = i$

• $\frac{1}{2+4i} = \frac{1}{2+4i}\,\frac{2-4i}{2-4i} = \frac{2-4i}{4+16} = \frac{2-4i}{20} = \frac{1}{10} - \frac{1}{5}i$

Practice stereographic projection: (Ex 3)

$P = (x,y,0) \mapsto (X,Y,Z)$

• (X,Y,Z) is on the sphere, so $X^2 + Y^2 + Z^2 = 1$

• and it's on the line $\ell = \overleftrightarrow{PN}$ $(N = (0,0,1))$

$\{tN + (1-t)P\} = \{((1-t)x, (1-t)y, t)\}$

so we must find t s.t.

$(1-t)^2x^2 + (1-t)^2y^2 + t^2 = 1$

i.e. $(1-t)^2x^2 + (1-t)^2y^2 = 1-t^2 = (1-t)(1+t)$

i.e. (since $t \neq 1$, we can divide by $1-t$)

$(1-t)(x^2+y^2) = (1+t)$

$\Leftrightarrow t(x^2+y^2+1) = x^2+y^2-1$

$\Leftrightarrow t = \frac{x^2+y^2-1}{x^2+y^2+1}$

So $Z = t = x^2+y^2-1 / x^2+y^2+1$

$X = (1-t)x = 2x / x^2+y^2+1$

$Y = (1-t)y = 2y / x^2+y^2+1.$