MATH 402 Homework 10

Due Friday November 30, 2018

Exercise 1.

a. Prove that in an omega triangle $PQ\Omega$ the sum of the angles at P and at Q is always strictly less that $180°$.

b. Prove *AA congruence* for omega triangles: that is suppose that you have two omega triangles $PQ\Omega$ and $P'Q'\Omega'$ with $\angle P \cong \angle P'$ and $\angle Q \cong \angle Q'$. Then show that $\overline{PQ} \cong \overline{P'Q'}$. *Hint: try a proof by contradiction. Remember that we proved SA congruence for omega triangles, and use your result from a.*

Exercise 2. In this exercise you will study what happens to omega points under reflections and rotations. Let ℓ be a hyperbolic line, and let r_ℓ be reflection across ℓ.

You may find it helpful to remember how reflections act (both in Euclidean and in hyperbolic geometry):

- If $P \in \ell$ then $r_\ell(P) = P$.
- If $P \notin \ell$ then r_ℓ is supposed to reflect P to the other side of the line ℓ. To find the image, draw the line m which passes through P and is perpendicular to ℓ; let Q be the point where m and ℓ intersect. Then $r_\ell(P)$ is the unique point of m on the other side of ℓ such that the distance from P to Q is the same as the distance from $r_\ell(P)$ to Q.

Also recall that on the last homework you proved that if m is limiting parallel to ℓ, then $r_\ell(m)$ is also limiting parallel to ℓ.

a. Let Ω be an omega point of ℓ. Show that r_ℓ fixes Ω. (What does it mean to *fix* Ω, given that Ω is a set?)

b. Now prove that the only omega points fixed by r_ℓ are the omega points of ℓ. Start by assuming that ℓ' is a line with different omega points. Suppose that r_ℓ fixes one of them, call it Ω'. Take P to be any point in ℓ, and consider the line $\overleftrightarrow{P\Omega'}$. What happens when you apply r_ℓ to this line? What if you took a different point $P' \in \ell$?

c. Suppose that $R_{A,\alpha}$ is rotation about some point A by angle α. Assume that $R_{A,\alpha}$ fixes an omega point Ω. Use what you've shown about reflections to prove that $\alpha = 0$ (i.e. $R_{A,\alpha}$ is the identity). *Hint: you can express the rotation as a composition of two reflections across lines which pass through A. Choose one of your lines so that you know what happens to Ω. What do you know about the second line?*

Exercise 3.

a. Prove that in a Saccheri quadrilateral, the summit is always larger than the base. *Hint: divide the Saccheri quadrilateral into two Lambert quadrilaterals, as we did in class. What do you know about opposite sides of a Lambert quadrilateral?*

b. Let $ABCD$ and $A'B'C'D'$ be two Saccheri quadrilaterals, with bases $\overline{AB}$ and $\overline{A'B'}$ and summits $\overline{CD}$ and $\overline{C'D'}$. Suppose that summits are congruent, and that the summit angles are also congruent. Prove that the Saccheri quadrilaterals themselves must be congruent: that is, prove that the bases are congruent, and that the sides are congruent.

   Hint: suppose (towards a contradiction) that the side $\overline{AD}$ is longer than the side $\overline{A'D'}$. Then you can find a point D'' on $\overline{AD}$ so that $AD'' = A'D'$ ... Try to construct a rectangle, which will give you the contradiction, because we know there are no rectangles in hyperbolic geometry.

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.