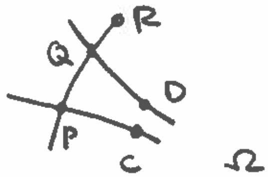
402 HW 10 Solutions.

Exercise 1 – (a) prove that in an omega triangle $PQ\Omega$, $\angle P + \angle Q < 180°$.



WTS $\angle CPQ + \angle DQP < 180°$.

Note: $\angle DQP + \angle DQR = 180°$

and by exterior angle theorem for Ω-triangles,

$\angle DQR > \angle CPQ$. so the result follows.

(b) Prove AA congruence for Ω-triangles

WTS $\overline{PQ} \cong \overline{P'Q'}$

Assume (for contradiction) that $\overline{PQ}$ is longer than $\overline{P'Q'}$ and choose $Q'' \in \overline{PQ}$ s.t. $\overline{PQ''} \cong \overline{P'Q'}$

Now by SA congruence for Ω-triangles, $PQ''\Omega \cong P'Q'\Omega'$

In particular, $\angle PQ''\Omega \cong \angle P'Q'\Omega'$, which we know is $\angle PQ\Omega$

$\Rightarrow$ $\angle QQ''\Omega$ is an omega triangle with angle sum $180°$ #.

Exercise 2:

(a) Ω an Ω-point of ℓ. show that $r_\ell(\Omega) = \Omega$.

(we need to show that $r_\ell(\Omega)$ and Ω are equal sets)

- if $m \in \Omega$, m is limiting parallel to ℓ at Ω, and by last week's HW, $r_\ell(m)$ is too. i.e. $r_\ell(m) \in \Omega$.

  $\Rightarrow r_\ell(\Omega) \subset \Omega$.

- conversely, given $m \in \Omega$ we want to show $m = r_\ell(m')$ for some $m' \in \Omega$, to show that $\Omega \subset r_\ell(\Omega)$.
  - by above, we know $r_\ell(m) \in \Omega$. so we can take $m' = r_\ell(m)$.

    $r_\ell(m') = r_\ell(r_\ell(m)) = m$.

$\therefore$ $r_\ell(\Omega) = \Omega$ as claimed.

b) Show that Ω-points of ℓ are the only Ω-points fixed by r_ℓ.

- Suppose towards a contradiction that Ω' is another Ω-point s.t. $r_\ell(\Omega') = \Omega'$.
- Choose a point P on ℓ, and let $m = \overleftrightarrow{P\Omega'} \neq \ell$.

  By assumption ~~Pm~~ $r_\ell(m) \in \Omega'$ again.

  but also $r_\ell(P) = P \in r_\ell(m)$, so $r_\ell(m) = \overleftrightarrow{P\Omega'} = m$.

  Now m is an invariant line of r_ℓ, and $m \neq \ell$, so we must have $m \perp \ell$.
- But if we choose $P' \neq P$ another point of ℓ, and let $m' = \overleftrightarrow{P'\Omega'}$, we again deduce that $m' \perp \ell$.

  So $PP'\Omega'$ is an omega triangle with two right angles.

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(c) Suppose $R_{A,\alpha}(\Omega) = \Omega$. Prove that $\alpha = 0$.

Let $\ell = \overleftrightarrow{A\Omega}$; choose m intersecting ℓ at A with angle $\alpha/2$

s.t. $R_{A,\alpha} = r_m \circ r_\ell$.

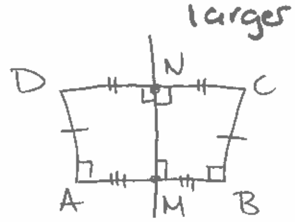
$$\left.\begin{array}{l} \text{Now } R_{A,\alpha}(\Omega) = \Omega \text{ by assumption} \\ r_m \circ r_\ell(\Omega) \underset{\substack{\uparrow \\ \text{by part (a)}}}{=} r_m(\Omega) \end{array}\right\} \Rightarrow r_m(\Omega) = \Omega.$$

By part (b), we see that $\Omega \in m$. so $\ell = m = \overleftrightarrow{A\Omega}$,

and $R_{A,\alpha} = id$

Exercise (3)

(a) Prove that in a Saccheri quadrilateral, the summit is always larger than the base.



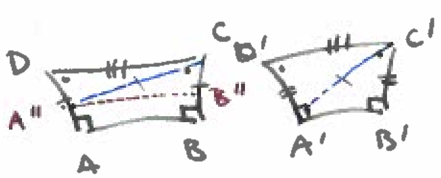
Let M be the midpoint of $\overline{AB}$; N the midpoint of $\overline{CD}$, so that $\overleftrightarrow{MN}$ is a common perpendicular as in the project.

Then NMBC and NMAD are Lambert quadrilaterals.

By theorem from class, $NC > MB$ and $ND > AM$.

$\Rightarrow CD = CN + ND > BM + MA = AB$

(b) Let ABCD and A'B'C'D' be two Saccheri quadrilaterals



Prove that they are congruent (i.e. their bases and sides are congruent).

Suppose towards a contradiction that $AD > A'D'$

Choose $A'' \in \overline{AD}$ and $B'' \in \overline{BC}$ s.t. $A''D = A'D'$, $B''C = B'C'$

Now by SAS $\Delta A''CD \cong \Delta A'C'D'$ and in particular $\overline{AC} \cong \overline{A'C'}$; also $\angle A''CD \cong \angle A'C'D'$ which implies $\angle A''CB'' \cong \angle A'C'B'$

So by SAS again, $\Delta A''CB'' \cong \Delta A'C'B'$ and in particular,

$$\left.\begin{array}{l} \angle CB''A'' = \angle C'B'A' = 90^\circ \\ \text{Likewise, } \angle DA''B'' = \angle D'A'B' = 90^\circ \end{array}\right\} \Rightarrow ABB''A'' \text{ is a rectangle.}$$

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