HW9 - MATH 402 - SOLUTIONS

Exercise 1

(a) Show that the symmetry group $S(P_n)$ of a regular n-gon is finite, by defining an injective function $\mathrm{Sym}(P_n) \to S_n$.

9 Let $V_1, \dots, V_n$ be the vertices of P_n.

Given $f \in \mathrm{Sym}(P_n)$, f permutes the vertices, giving a function

$$\mathrm{Sym}(P_n) \longrightarrow \{\text{permutations of } V_1, \dots, V_n\} = S_n$$

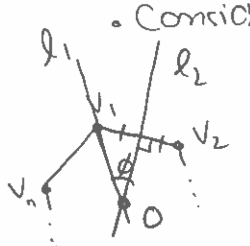
This function is injective: if f, g give the same permutation, they agree on V_1, V_2, V_3, three non-collinear points.

$\Rightarrow$ the number of elements in $\mathrm{Sym}(P_n)$ is $\leq$ the number of elements in S_n, which is $n!$.

(b) Show that $\mathrm{Sym}(P_n) = D_n$.

15 Since $\mathrm{Sym}(P_n)$ is a finite planar symmetry group, we can identify it by finding the smallest rotation $R_{O,\alpha}$, and by determining whether it has any reflections.

- we already know that it does have reflections (across angle bisectors or perpendicular bisectors of sides)
- Note that the smallest rotation will be one of maximal order. Since a rotation shifts the vertices cyclically and there are n vertices, the order of a rotation is at most n. So if we can find a rotation of order n, we will know that there are exactly n rotations in $\mathrm{Sym}(P_n)$, and hence that $\mathrm{Sym}(P_n) = D_n$.
- Consider the adjacent vertices V_n, V_1, V_2.



 - let l_1 be the angle bisector of $\angle V_n V_1 V_2$.
 - let l_2 be the perpendicular bisector of side $\overline{V_1 V_2}$
 - Since V_n, V_1, V_2 are not collinear, l_1 and l_2 are not parallel, and they intersect at a point O, forming an angle ϕ
- One can see that $\Delta V_1 O V_2$ has angle 2ϕ at V_1 and that for any i, $\Delta V_i O V_{i+1}$ is congruent (e.g. by ASA).

$\Rightarrow n(2\phi) = 360$, so $2\phi = 360/n$.

and $r_{\ell_1} \circ r_{\ell_2} = Rot_{O, 2\phi}$ has order n.

$\therefore$ $Sym(P_n) = D_n$ as claimed. $\square$.

(c) Draw a figure whose symmetry group is ~~either~~ cyclic, not dihedral.

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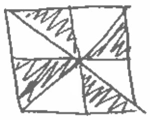
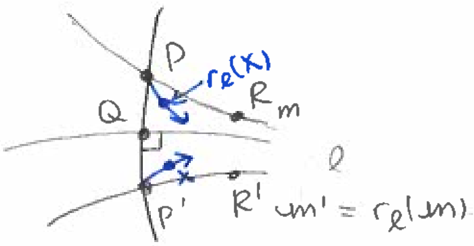


Exercise 3. Let ℓ be a hyperbolic line and let r_ℓ be reflection across ℓ.

13 Suppose m is limiting parallel to ℓ ~~at~~

Prove that $r_\ell(m)$ is limiting parallel to ℓ too.



Choose $P \in m$, so m is limiting parallel to ℓ at P.

let $m' = r_\ell(m)$, $P' = r_\ell(P) \in m'$.

We claim m' is limiting parallel to ℓ at P'.

- $m \parallel \ell \Rightarrow r_\ell(m) \parallel r_\ell(\ell) = \ell$ i.e. $m' \parallel \ell$.

- Since $P' = r_\ell(P)$, $\overleftrightarrow{PP'} \perp \ell$; call the intersection point Q

We need to show that any ray $\overrightarrow{P'X}$ interior to $\angle QP'R'$ intersects ℓ

Consider instead $r_\ell(\overrightarrow{P'X}) = \overrightarrow{P r_\ell(X)}$, interior to $\angle QPR$.

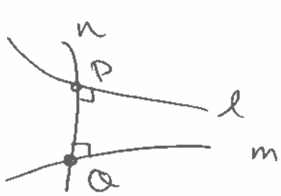
Since m is limiting parallel to ℓ at P, $\overrightarrow{P r_\ell(X)}$ intersects ℓ at some point T.

$\Rightarrow r_\ell(\overrightarrow{P r_\ell(X)}) = \overrightarrow{P'X}$ intersects $r_\ell(\ell) = \ell$ at $r_\ell(T) = T$.

Exercise 2. Let ℓ and m be limiting parallel.

Prove that they have no common perpendicular.

8 Suppose towards a contradiction that $n \perp \ell$, $n \perp m$.



- then ℓ is limiting parallel to m at P with angle of parallelism $\alpha = 90$.

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$\square$