Math 402 - HW 6 solutions.

Ex. 1: Suppose f, g are isometries which agree on three non-collinear points A, B, C. Show that $f = g$.

Consider $f^{-1} \circ g$: $f^{-1} \circ g(A) = f^{-1} \circ f(A) = A$

and likewise $f^{-1} \circ g(B) = B$

$f^{-1} \circ g(C) = C$.

So $f^{-1} \circ g$ is an isometry with three non-collinear fixed points

$\therefore f^{-1} \circ g = id$

$\Rightarrow f = g$.

Ex. 2: Prove that an isometry preserves circles.

let $c = \{ X \mid OX = r \}$

$f(c) = \{ f(X) \mid OX = r \} = \{ f(X) \mid f(O)f(X) = r \}$

because $f(O)f(X) = OX$.

$\Rightarrow f(C)$ is the circle with centre $f(O)$ and radius r.

Ex. 3: (a) define what it means for a set S to be fixed or invariant under r.

- S is fixed if $\forall P \in S$, $r(P) = P$
- S is invariant if $\forall P \in S$, $r(P) \in S$.

(b) prove that the invariant lines of r_ℓ are exactly ℓ and the lines perpendicular to ℓ.

- ℓ is fixed $\Rightarrow$ ℓ is invariant
- we need to prove that $m \neq \ell$ is invariant $\Leftrightarrow$ $m \perp \ell$.

" $\Rightarrow$ " $r(m) \subset m$.

take $P \in m$, $P \notin \ell$. then $r(P) \neq P$, but by assumption $r(P) \in m$

so $m = \overleftrightarrow{P r(P)}$, and by the result in class, ℓ is the perpendicular bisector of $\overline{P r(P)}$.

in particular $m \perp \ell$.

" $\Leftarrow$ " if $m \perp \ell$ and $P \in m$, we want to show $r(P) \in m$.

- if $P \in \ell$, $r(P) = P \in m$.
- if $P \notin \ell$, we know ℓ is the perpendicular bisector of $\overline{P r(P)}$, so $r(P)$ is on the line m.

Ex. 4. (a). $T = r_{\ell_2} \circ r_{\ell_1}$ with displacement vector v. show T^{-1} is a translation, with $T^{-1} = r_{\ell_1} \circ r_{\ell_2}$, translation vector $\vec{v}$.

<u>Method 1:</u> in a coordinate system s.t. $\vec{v} = \overrightarrow{(v_1, v_2)}$, T is given by

$T(x,y) = (x,y) + (v_1, v_2)$.

let $S(x,y) = (x,y) - (v_1, v_2)$ — by theorem, it is a translation with translation vector $-\vec{v}$.

It is easy to see that $T \circ S = id$, $S \circ T = id$, so

$S = T^{-1}$.

But we also know that $T^{-1} = (r_{\ell_2} \circ r_{\ell_1})^{-1} = r_{\ell_1}^{-1} \circ r_{\ell_2}^{-1}$

$= r_{\ell_1} \circ r_{\ell_2}$

<u>Method 2:</u> As above, $T^{-1} = r_{\ell_1} \circ r_{\ell_2}$, and since $\ell_1 // \ell_2$ (or $\ell_1 = \ell_2$) this is a translation.

To find the translation vector $\vec{w}$ take any point A and look at the vector $\overrightarrow{A T^{-1}(A)} =: \vec{w}$

Choose $A = T(B)$ for some point B.

so $\vec{w} = \overrightarrow{T(B) B}$. But $\overrightarrow{B T(B)} = \vec{v}$, so

$\overrightarrow{T(B) B} = -\vec{v}$.

(b) Let T_1, T_2 be translations with displacement vectors $\vec{v}_1, \vec{v}_2$.
What is $T_1 \circ T_2$?

in coordinates s.t. $\vec{v}_1 = (a_1, a_2)$, $\vec{v}_2 = (b_1, b_2)$

$$\begin{cases} T_1(x,y) = (x,y) + (a_1,a_2) = (x+a_1, y+a_2) \\ T_2(x,y) = (x+b_1, y+b_2) \end{cases}$$

$$\Rightarrow T_1 \circ T_2(x,y) = (x+a_1+b_1, y+a_2+b_2)$$
$$= (x,y) + (a_1+b_1, a_2+b_2)$$

By theorem, this is a translation with translation vector

$$\overrightarrow{(a_1+b_1, a_2+b_2)} = \vec{v}_1 + \vec{v}_2$$

(c) Show that composition of translations commutes.
Is this true for reflections?

By (b), $T_1 \circ T_2$ = translation by $\vec{v}_1 + \vec{v}_2$

$T_2 \circ T_1$ = translation by $\vec{v}_2 + \vec{v}_1$

But $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$, so $T_1 \circ T_2 = T_2 \circ T_1$.

On the other hand, by part (a), we see that for $\ell_1 \parallel \ell_2$, $r_{\ell_1} \circ r_{\ell_2}$ and $r_{\ell_2} \circ r_{\ell_1}$ give translations in opposite directions, so $r_{\ell_1} \circ r_{\ell_2} \neq r_{\ell_2} \circ r_{\ell_1}$.

(d) Does the set of translations form a group?

Yes.

- By (b) the composition of two translations is a translation.
- Composition is always associative.
- By definition, $id = r_\ell \circ r_\ell$ is a translation
- By (a), the inverse of a translation is again a translation.