Factorisation algebras associated to Hilbert schemes of points

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Motivation

- **Learn about factorisation:**
  Provide and study examples of factorisation spaces and algebras of arbitrary dimensions.
- **Learn about Hilbert schemes:**
  Factorisation structures formalise the intuition that a space is built out of local bits in a specific way.
  Factorisation structures are expected to arise, based on the work of Grojnowski and Nakajima.

Outline

1. Main constructions : $\mathcal{H}ilb_{\mathrm{Ran}\,X}$ and $\mathcal{H}_{\mathrm{Ran}\,X}$
2. Chiral algebras
3. Results on $\mathcal{H}_{\mathrm{Ran}\,X}$

Section 1

Main constructions : $\mathcal{H}ilb_{\mathrm{Ran}\,X}$ and $\mathcal{H}_{\mathrm{Ran}\,X}$

Notation

- Fix k an algebraically closed field of characteristic 0.
- Let X be a smooth variety over k of dimension d.
- We work in the category of prestacks:

$$\begin{array}{ll} \mathrm{PreStk} & := \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \infty\text{-Grpd}) \\ \uparrow & \\ \mathrm{Sch} & \text{(Yoneda embedding)} \end{array}$$

The Hilbert scheme of points

Fix $n \geq 0$. The Hilbert scheme of n points in X is (the scheme representing) the functor

$$\begin{aligned} \mathrm{Hilb}_X^n : \mathrm{Sch}^{\mathrm{op}} &\to \mathrm{Set} \subset \infty\text{-}\mathrm{Grpd} \\ S &\mapsto \mathrm{Hilb}_X^n(S), \end{aligned}$$

where

$$\mathrm{Hilb}_X^n(S) := \left\{ \begin{array}{l} \xi \subset S \times X, \text{ a closed subscheme, flat over } S \\ \text{with zero-dimensional fibres of length } n \end{array} \right\}.$$

The Hilbert scheme of points

Example: k-points

$$\mathrm{Hilb}^n_X(\operatorname{Spec} k) = \left\{ \begin{array}{c} \xi \subset X \text{ closed zero-dimensional} \\ \text{subscheme of length } n \end{array} \right\}.$$

For example, for $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, $n = 2$, some k-points are

$$\begin{aligned} \xi_1 &= \operatorname{Spec} k[x, y]/(x, y^2) \\ \xi_2 &= \operatorname{Spec} k[x, y]/(x^2, y) \\ \xi_3 &= \operatorname{Spec} k[x, y]/(x, y(y-1)). \end{aligned}$$

Notation: let $\mathrm{Hilb}_X := \bigsqcup_{n \geq 0} \mathrm{Hilb}^n_X$.

The Ran space

The Ran space is a different way of parametrising sets of points in X:

$$\operatorname{Ran} X(S) := \{A \subset \operatorname{Hom}(S, X), \text{ a finite, non-empty set }\}.$$

Let $A = \{x_1, \ldots, x_d | \ x_i : S \to X\}$ be an S-point of $\operatorname{Ran} X$.

For each x_i, let $\Gamma_{x_i} = \{(s, x_i(s)) \in S \times X\}$ be its graph, and define

$$\Gamma_A := \bigcup_{i=1}^{d} \Gamma_{x_i} \subset S \times X,$$

a closed subscheme with the reduced scheme structure.

The Ran space

The Ran space is not representable by a scheme, but it is a pseudo-indscheme:

$$\operatorname{Ran} X = \operatorname*{colim}_{I \in \mathrm{fSet}^{\mathrm{op}}} X^I.$$

Here the colimit is taken in PreStk, over the closed diagonal embeddings

$$\Delta(\alpha) : X^J \hookrightarrow X^I$$

induced by surjections of finite sets

$$\alpha : I \twoheadrightarrow J.$$

Main definition: $\mathcal{H}ilb_{\operatorname{Ran} X}$

Define the prestack

$$\begin{aligned} \mathcal{H}ilb_{\operatorname{Ran} X} : \mathrm{Sch}^{\mathrm{op}} &\to \mathrm{Set} \subset \infty\text{-Grpd} \\ S &\mapsto \mathcal{H}ilb_{\operatorname{Ran} X}(S) \end{aligned}$$

by setting $\mathcal{H}ilb_{\operatorname{Ran} X}(S)$ to be the set

$$\{(A, \xi) \in (\operatorname{Ran} X \times \mathrm{Hilb}_X)(S) \mid \operatorname{Supp}(\xi) \subset \Gamma_A \subset S \times X\}.$$

Note: This is a *set-theoretic* condition.

Notation: We have natural projection maps

$$\begin{aligned} f &: \mathcal{H}ilb_{\operatorname{Ran} X} \to \operatorname{Ran} X, \\ \rho &: \mathcal{H}ilb_{\operatorname{Ran} X} \to \mathrm{Hilb}_X . \end{aligned}$$

$\mathcal{H}ilb_{\mathrm{Ran}\, X}$ as a pseudo-indscheme

For a finite set I, we define

$$\mathcal{H}ilb_{X^I} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$$

by setting $\mathcal{H}ilb_{X^I}(S) \subset (X^I \times \mathrm{Hilb}_X)(S)$ to be

$$\left\{ ((x_i)_{i \in I}, \xi) \mid (\{x_i\}_{i \in i}, \xi) \in \mathcal{H}ilb_{\mathrm{Ran}\, X}(S) \right\}.$$

For $\alpha : I \twoheadrightarrow J$, we have natural maps

$$\mathcal{H}ilb_{X^J} \to \mathcal{H}ilb_{X^I},$$

defined by $((x_j)_{j \in J}, \xi) \mapsto (\Delta(\alpha)(x_j), \xi)$.

Then $\mathcal{H}ilb_{\mathrm{Ran}\, X} = \operatorname*{colim}_{I \in \mathrm{fSet}^{\mathrm{op}}} \mathcal{H}ilb_{X^I}$.

Factorisation

Consider $(\mathcal{H}ilb_{\operatorname{Ran} X})_{\mathrm{disj}} = \{(A = A_1 \sqcup A_2, \xi) \in \mathcal{H}ilb_{\operatorname{Ran} X}\}$.

Suppose that in fact $\Gamma_{A_1} \cap \Gamma_{A_2} = \emptyset$, so that if we set $\xi_i := \xi \cap \widehat{\Gamma}_{A_i}$, we see that

1. $\xi = \xi_1 \sqcup \xi_2$
2. $(A_i, \xi_i) \in \mathcal{H}ilb_{\operatorname{Ran} X}$ for $i = 1, 2$.

Proposition

$$(\mathcal{H}ilb_{\operatorname{Ran} X})_{disj} \simeq (\mathcal{H}ilb_{\operatorname{Ran} X} \times \mathcal{H}ilb_{\operatorname{Ran} X})_{disj}.$$

Factorisation

In particular, when $A = \{x_1\} \sqcup \{x_2\}$, we can express this formally as follows:

- Set $U := X^2 \setminus \Delta(X) \xhookrightarrow{j} X^2$.
- Then the proposition specialises to the statement that there exists a canonical isomorphism

$$c : \mathcal{H}ilb_{X^2} \times_{X^2} U \xrightarrow{\simeq} (\mathcal{H}ilb_X \times \mathcal{H}ilb_X) \times_{X \times X} U.$$

We have similar isomorphisms $c(\alpha)$ associated to any surjection of finite sets $I \twoheadrightarrow J$. These are called factorisation isomorphisms.

Factorisation

Theorem

$f : \mathcal{H}ilb_{\mathrm{Ran}\,X} \to \mathrm{Ran}\,X$ defines a factorisation space on X. If X is proper, f is an ind-proper morphism.

Linearisation of $\mathcal{H}ilb_{\mathrm{Ran}\, X}$

Set-up: Let $\lambda^I \in \mathcal{D}(\mathcal{H}ilb_{X^I})$ be a family of (complexes of) $\mathcal{D}$-modules compatible with the factorisation structure.

Then the family $\left\{\mathcal{A}_{X^I} := (f_I)_! \lambda^I \in \mathcal{D}(X^I)\right\}$ defines a factorisation algebra on X.

More precisely: For every $\alpha : I = \bigsqcup_{j \in J} I_j \twoheadrightarrow J$, we have isomorphisms

1. $v(\alpha) : \Delta(\alpha)^! \mathcal{A}_{X^I} \xrightarrow{\sim} \mathcal{A}_{X^J}$

   $\Rightarrow \{\mathcal{A}_{X^I}\}$ give an object "$\operatorname{colim} \mathcal{A}_{X^I}$ " of $\mathcal{D}(\mathrm{Ran}\, X)$, which we'll denote by $f_! \lambda$.

2. $c(\alpha) : j(\alpha)^* (\mathcal{A}_{X^I}) \xrightarrow{\sim} j(\alpha)^* \left(\boxtimes_{j \in J} \mathcal{A}_{X^{I_j}}\right)$

Linearisation of $\mathcal{H}ilb_{\operatorname{Ran} X}$

Definition

Set $\mathcal{H}_{X^I} := (f_I)_! \omega_{\mathcal{H}ilb_{X^I}}$.

This gives a factorisation algebra

$$\mathcal{H}_{\operatorname{Ran} X} = f_! \omega_{\mathcal{H}ilb_{\operatorname{Ran} X}}.$$

Goal for the rest of the talk: study this factorisation algebra.

Section 2

Chiral algebras

Chiral algebras

A chiral algebra on X is a $\mathcal{D}$-module $\mathcal{A}_X$ on X equipped with a Lie bracket

$$\mu_{\mathcal{A}} : j_* j^* \left(\mathcal{A}_X \boxtimes \mathcal{A}_X \right) \to \Delta_! \mathcal{A}_X \in \mathcal{D}(X \times X).$$

Factorisation algebras and chiral algebras

Theorem (Beilinson–Drinfeld, Francis–Gaitsgory)

We have an equivalence of categories

$$\left\{ \begin{array}{c} \textit{factorisation algebras} \\ \textit{on } X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{chiral algebras} \\ \textit{on } X \end{array} \right\}.$$

Idea of the proof

Let $\{\mathcal{A}_{X^I}\}$ be a factorisation algebra.

$$\begin{array}{ccccc} & & j_*j^*(\mathcal{A}_X \boxtimes \mathcal{A}_X) & & \\ & & \downarrow \wr & & \\ \mathcal{A}_{X^2} & \longrightarrow & j_*j^*(\mathcal{A}_{X^2}) & \longrightarrow & \Delta_!\Delta^!\mathcal{A}_{X^2} \\ & & & & \downarrow \wr \\ & & & & \Delta_!\mathcal{A}_X \end{array}$$

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This defines $\mu_{\mathcal{A}} : j_* j^* (\mathcal{A}_X \boxtimes \mathcal{A}_X) \to \Delta_! \mathcal{A}_X$.

To check the Jacobi identity, we use the factorisation isomorphisms for $I = \{1, 2, 3\}$.

Aside: chiral algebras and vertex algebras

Let $(V, Y(\cdot, z), |0\rangle)$ be a quasi-conformal vertex algebra, and let C be a smooth curve.

We can use this data to construct a chiral algebra $(\mathcal{V}_C, \mu)$ on C.

This procedure works for any smooth curve C, and gives a compatible family of chiral algebras. Together, all of these chiral algebras form a universal chiral algebra of dimension 1.

Lie $\star$ algebras

A Lie $\star$ algebra on X is a $\mathcal{D}$-module $\mathcal{L}$ on X equipped with a Lie bracket

$$\mathcal{L} \boxtimes \mathcal{L} \to \Delta_! \mathcal{L}.$$

Example: we have a canonical embedding

$$\mathcal{A}_X \boxtimes \mathcal{A}_X \to j_* j^* \left(\mathcal{A}_X \boxtimes \mathcal{A}_X \right).$$

So every chiral algebra $\mathcal{A}_X$ is a Lie $\star$ algebra.

Universal chiral enveloping algebras

The resulting forgetful functor

$$F : \{\text{chiral algebras}\} \to \{\text{Lie} \star \text{algebras}\}$$

has a left adjoint

$$U^{\mathrm{ch}} : \{\text{Lie} \star \text{algebras}\} \to \{\text{chiral algebras}\}\,.$$

$U^{\mathrm{ch}}(\mathcal{L})$ is the universal chiral envelope of $\mathcal{L}$.

1. $U^{\mathrm{ch}}(\mathcal{L})$ has a natural filtration, and there is a version of the PBW theorem.
2. $U^{\mathrm{ch}}(\mathcal{L})$ has a structure of chiral Hopf algebra.

Commutative chiral algebras

A chiral algebra $\mathcal{A}_X$ is commutative if the underlying Lie $\star$ bracket is zero.

Translation into factorisation language:

$$\begin{array}{ccccc} & & j_*j^*(\mathcal{A}_X \boxtimes \mathcal{A}_X) & & \\ & & \downarrow \wr & & \\ \mathcal{A}_{X^2} & \longrightarrow & j_*j^*(\mathcal{A}_{X^2}) & \longrightarrow & \Delta_! \Delta^! \mathcal{A}_{X^2} \\ & & & & \downarrow \wr \\ & & & & \Delta_! \mathcal{A}_X \end{array}$$

Commutative chiral algebras

A chiral algebra $\mathcal{A}_X$ is commutative if the underlying Lie $\star$ bracket is zero.

Translation into factorisation language:

$$
\begin{array}{ccccc}
\mathcal{A}_X \boxtimes \mathcal{A}_X & \hookrightarrow & j_*j^*(\mathcal{A}_X \boxtimes \mathcal{A}_X) & & \\
 & & \downarrow \wr & & \\
\mathcal{A}_{X^2} & \longrightarrow & j_*j^*(\mathcal{A}_{X^2}) & \longrightarrow & \Delta_!\Delta^!\mathcal{A}_{X^2} \\
 & & & 0 & \downarrow \wr \\
 & & & & \Delta_!\mathcal{A}_X
\end{array}
$$

Commutative chiral algebras

A chiral algebra $\mathcal{A}_X$ is commutative if the underlying Lie $\star$ bracket is zero.

Translation into factorisation language:

$$
\begin{array}{ccccc}
\mathcal{A}_X \boxtimes \mathcal{A}_X & \longrightarrow & j_* j^* (\mathcal{A}_X \boxtimes \mathcal{A}_X) & & \\
\downarrow & & \downarrow \wr & & \\
\mathcal{A}_{X^2} & \longrightarrow & j_* j^* (\mathcal{A}_{X^2}) & \longrightarrow & \Delta_! \Delta^! \mathcal{A}_{X^2} \\
 & & & & \downarrow \wr \\
 & & & & \Delta_! \mathcal{A}_X
\end{array}
$$

Commutative factorisation algebras

A factorisation algebra $\{\mathcal{A}_{X^I}\}$ is commutative if every factorisation isomorphism

$$c(\alpha)^{-1} : j^* \left(\boxtimes_{j \in J} \mathcal{A}_{X^{I_j}}\right) \xrightarrow{\simeq} j^* \mathcal{A}_{X^I}$$

extends to a map of $\mathcal{D}$-modules on all of X^I:

$$\boxtimes_{j \in J} \mathcal{A}_{X^{I_j}} \to \mathcal{A}_{X^I}.$$

Proposition (Beilinson–Drinfeld)

We have equivalences of categories

$$\left\{ \begin{array}{c} \textit{commuative} \\ \textit{factorisation} \\ \textit{algebras} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{commutative} \\ \textit{chiral} \\ \textit{algebras} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{commutative} \\ \mathcal{D}_X\textit{-algebras} \end{array} \right\}.$$

Section 3

Results on $\mathcal{H}_{\mathrm{Ran}\,X}$

Chiral homology

Let $p_{\operatorname{Ran} X} : \operatorname{Ran} X \to \mathrm{pt}$.

The chiral homology of a factorisation algebra $\mathcal{A}_{\operatorname{Ran} X}$ is defined by

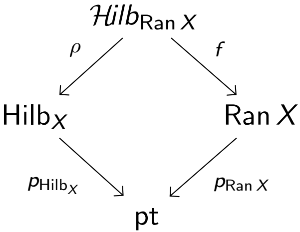
$$\int \mathcal{A}_{\operatorname{Ran} X} := p_{RanX,!}\mathcal{A}_{\operatorname{Ran} X}.$$

It is a derived formulation of the space of conformal blocks of a vertex algebra V:

$$H^0(\int \mathcal{V}_{\operatorname{Ran} X}) = \text{space of conformal blocks of } V.$$

The chiral homology of $\mathcal{H}_{\operatorname{Ran} X}$

Goal: compute $\int \mathcal{H}_{\operatorname{Ran} X} := p_{\operatorname{Ran} X,!} f_! \omega_{\mathcal{H}ilb_{\operatorname{Ran} X}}$.



$$\Rightarrow \int \mathcal{H}_{\operatorname{Ran} X} \simeq p_{\operatorname{Hilb}_X,!} \rho_! \omega_{\mathcal{H}ilb_{\operatorname{Ran} X}}$$
$$\simeq p_{\operatorname{Hilb}_X,!} \rho_! \rho^! \omega_{\operatorname{Hilb}_X}.$$

The chiral homology of $\mathcal{H}_{\operatorname{Ran} X}$

Theorem

$$\rho^! : \mathcal{D}(\mathrm{Hilb}_X) \to \mathcal{D}(\mathrm{Hilb}_{\operatorname{Ran} X})$$

is fully faithful, and hence $\rho_! \circ \rho^! \to \mathrm{id}_{\mathcal{D}(\mathrm{Hilb}_X)}$ *is an equivalence.*

Corollary

$$\int \mathcal{H}_{\operatorname{Ran} X} \simeq p_{\mathrm{Hilb}_X,!}\omega_{\mathrm{Hilb}_X} := H^\bullet_{dR}(\mathrm{Hilb}_X).$$

Identifying the factorisation algebra structure on $\mathcal{H}_{\operatorname{Ran} X}$

Theorem

The assignment

$$\underset{\textit{dim. } d}{X} \rightsquigarrow \mathcal{H}_{\operatorname{Ran} X}$$

gives rise to a universal factorisation algebra of dimension d.

i.e. it behaves well in families, and is compatible under pullback by étale morphisms $Y \to X$.

This allows us to reduce to the study of $\mathcal{H}_{\operatorname{Ran} X}$ for $X = \mathbb{A}^d = \operatorname{Spec} k[x_1, \ldots, x_d]$.

Identifying the factorisation algebra structure on $\mathcal{H}_{\mathrm{Ran}\,\mathbb{A}^d}$

Conjecture

$\mathcal{H}_{\mathrm{Ran}\,\mathbb{A}^d}$ is a commutative factorisation algebra.

Remarks on the proof:

1. The case $d = 1$ is clear:
   $\mathcal{H}ilb_{\mathrm{Ran}\,\mathbb{A}^1}$ is a commutative factorisation space.
2. The case $d = 2$ has been proven by Kotov using deformation theory.

Strategy for general d: first step

The choice of a global coordinate system $\{x_1, \ldots, x_d\}$ gives an identification of

$$\mathrm{Hilb}_{X,0} := \{\xi \in \mathrm{Hilb}_X \mid \mathrm{Supp}(\xi) = \{0\}\}$$

with $\mathrm{Hilb}_{X,p}$ for every $p \in X = \mathbb{A}^d$.

$$\Rightarrow \mathcal{H}ilb_X \simeq X \times \mathrm{Hilb}_{X,0}\,.$$

It follows that

$$\mathcal{H}_X \simeq \omega_X \otimes H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0}).$$

Strategy for general d: second step

Universality of $\mathcal{H}_{\mathrm{Ran}_\bullet}$ means that, in particular, the fibre of $\mathcal{H}_{\mathbb{A}^d}$ over $0 \in \mathbb{A}^d$, is a representation of the group

$$G = \underline{\mathrm{Aut}} k[\![t_1, \ldots, t_d]\!].$$

This fibre is $H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0})$, and the representation is induced from the action of G on the space $\mathrm{Hilb}_{X,0}$.

Strategy for general d: steps 3, 4 ...

Claim 1: The induced action is canonically trivial, except perhaps for an action of $\mathbb{G}_m \subset G$ corresponding to a grading.

Claim 2: This forces the chiral bracket

$$j_* j^*(\omega_X \boxtimes \omega_X) \otimes H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0}) \otimes H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0}) \\ \to \Delta_!(\omega_X) \otimes H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0})$$

to be of the form $\mu_{\omega_X} \otimes m$, where m is a map

$$H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0}) \otimes H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0}) \to H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0}).$$

Claim 3: m induces a commutative $\mathcal{D}_X$-algebra structure on $\mathcal{H}_X = \omega_X \otimes H^\bullet_{\mathrm{dR}}(\mathrm{Hilb}_{X,0})$.

Claims 1 and 2 seem straightforward to prove in the non-derived setting, but in the derived setting there are subtleties.

Future directions

- Push forward other sheaves to get more interesting factorisation algebras: replace $\omega_{\mathcal{H}ilb_{X^I}}$ by sheaves constructed from e.g. tautological bundles, sheaves of vanishing cycles.
- How is this related to the work of Nakajima and Grojnowski?