Recorências e difusão anômala em sistemas Hamiltoneanos caóticos

IFUSP - Março 2010

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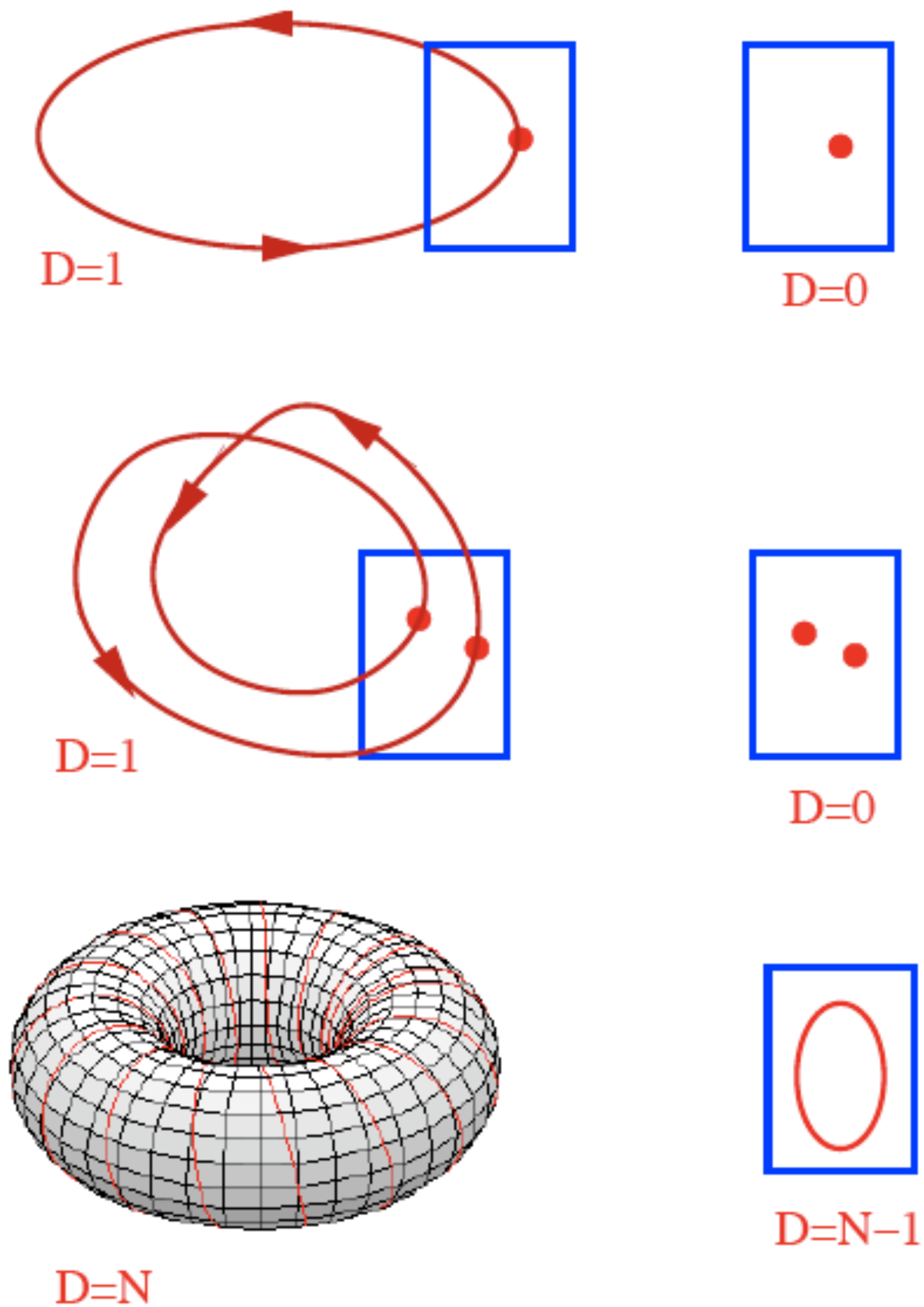
http://www.tinyurl.com/ifusp2010

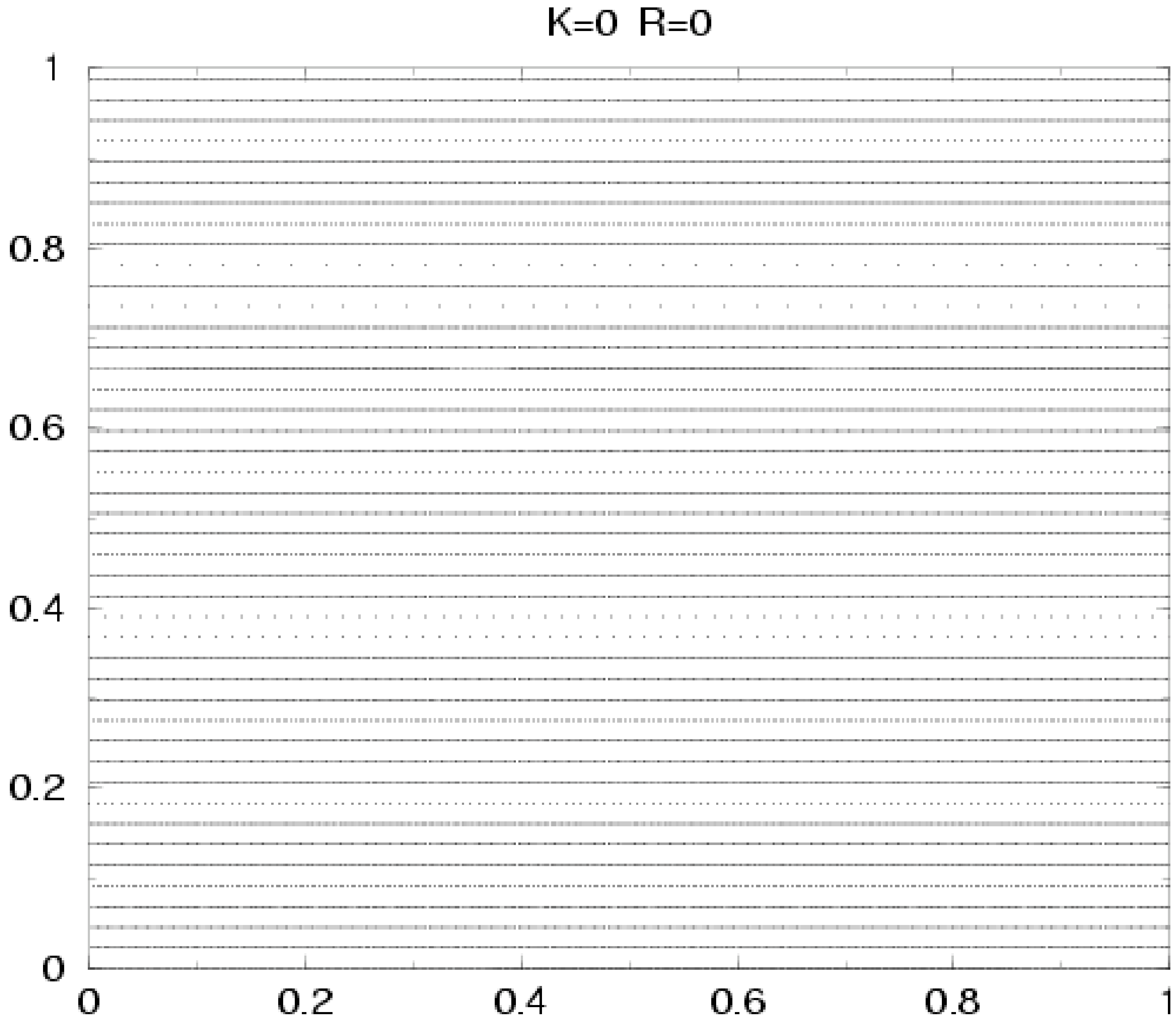
Apresentação 1:

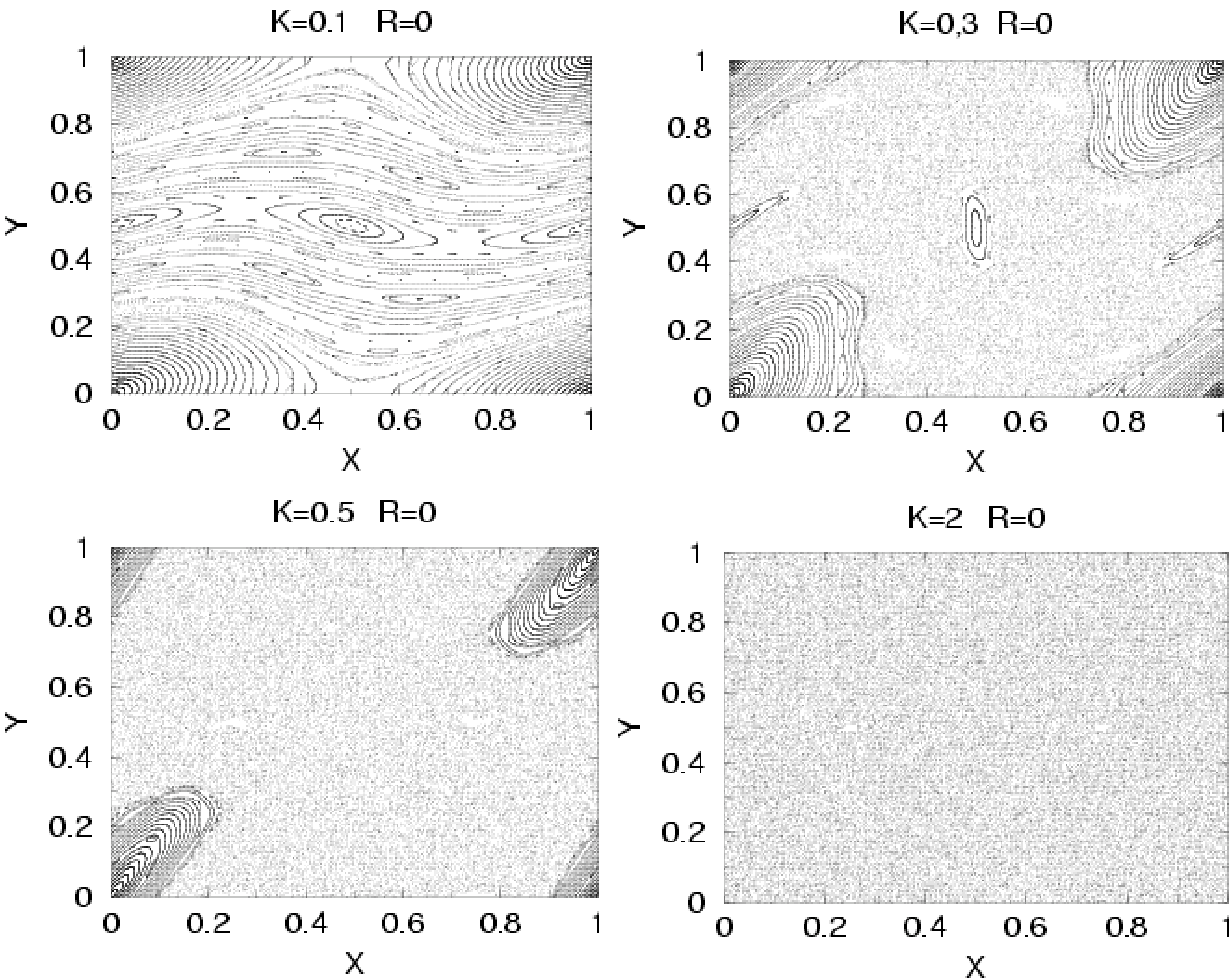
Torus, mapa padrão, ilhas ao redor de ilhas



D=1
D=0
D=1
D=0
D=N
D=N−1

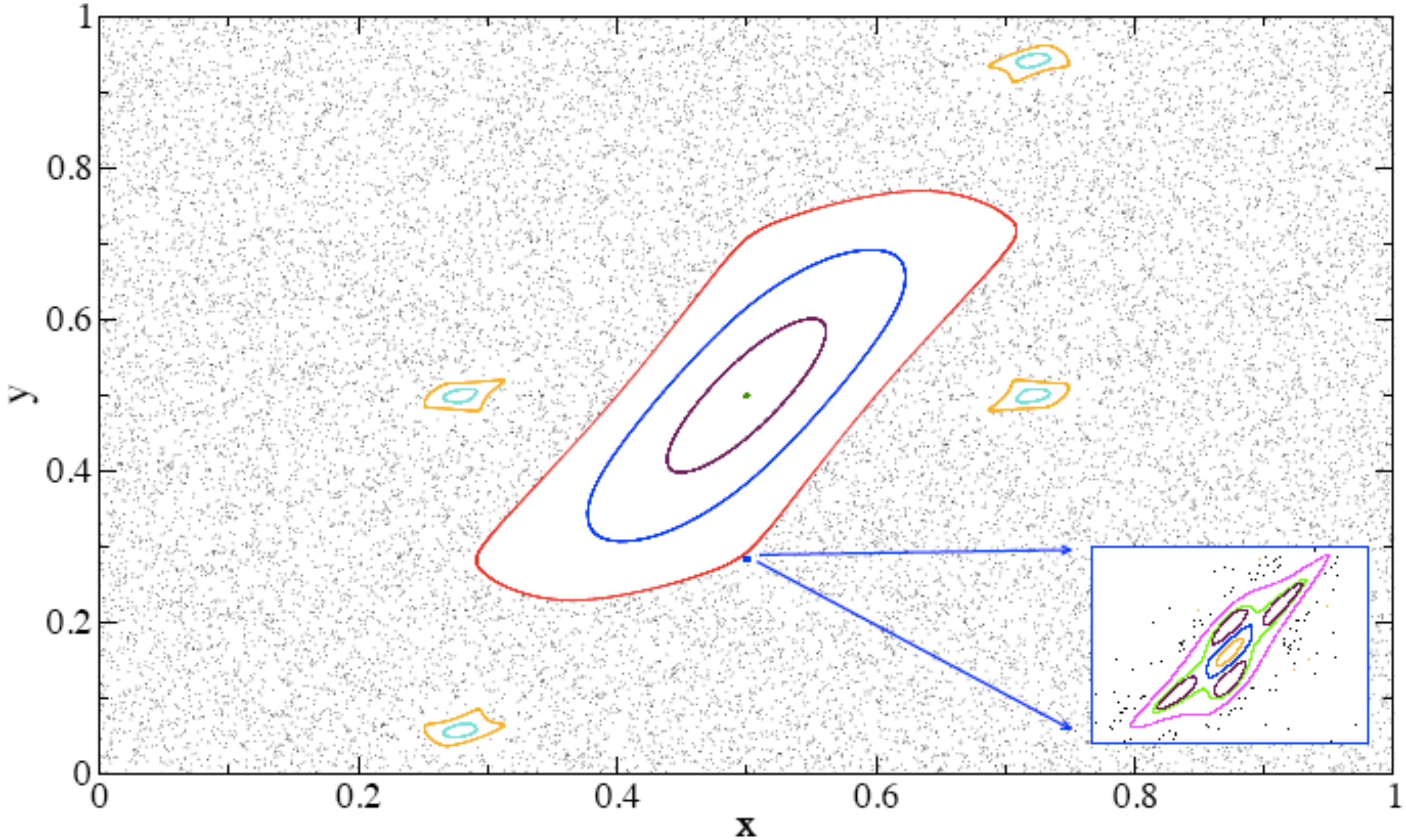
$$y_{i+1} = y_i + K \sin(2\pi x_i) \quad \text{mod } 1,$$
$$x_{i+1} = x_i + y_{i+1} \quad \text{mod } 1,$$





K=0.1 R=0
K=0,3 R=0
K=0.5 R=0
K=2 R=0
X
Y
0
0.2
0.4
0.6
0.8
1

$K = 0.45$



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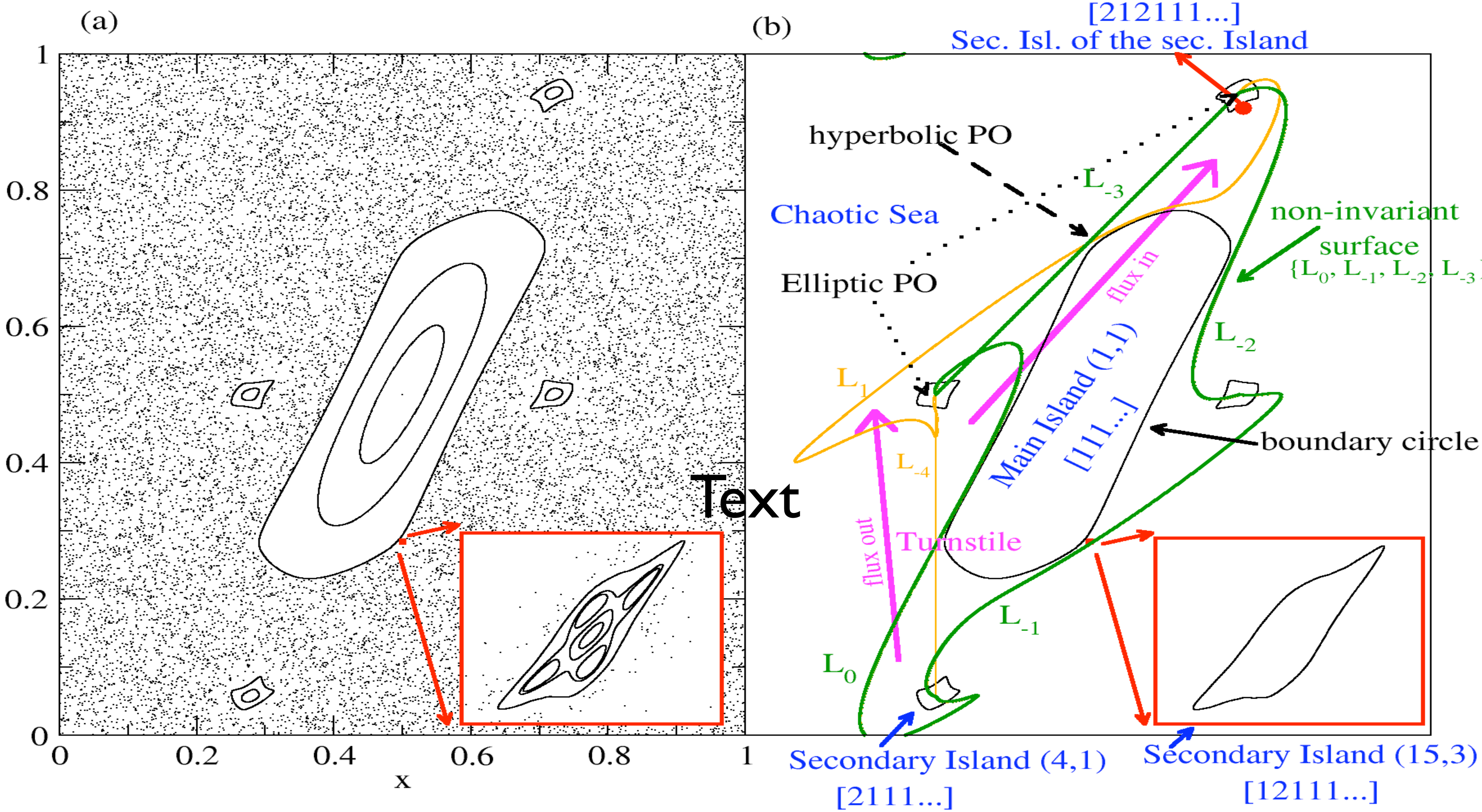


(a)
(b)
x
[212111...]
Sec. Isl. of the sec. Island
hyperbolic PO
Chaotic Sea
Elliptic PO
non-invariant surface
{L_0, L_-1, L_-2, L_-3}
flux in
Main Island (1,1)
[111...]
boundary circle
Text
flux out
Turnstile
Secondary Island (4,1)
[2111...]
Secondary Island (15,3)
[12111...]

Apresentação II:

Recorrências para detectar rompimento de tori

PHYSICAL REVIEW E **73**, 056201 (2006)

Nontwist non-Hamiltonian systems

E. G. Altmann,* G. Cristadoro,† and D. Pazó‡

Interesse em sistemas temporalmente reversíveis (dinâmica quasi-Hamiltoneana pode aparecer)

$$\frac{d(G\mathbf{x})}{dt} = -\mathbf{F}(G(\mathbf{x})) \quad \text{and} \quad L \circ G\mathbf{x}_{n+1} = G\mathbf{x}_n,$$

Se mais de uma simetria esta presente no sistema a condição de torção (twist) é violada:

$$\det\left|\frac{\partial \dot{\theta}_k}{\partial I_j}\right| \neq 0 \quad \text{and} \quad \det\left|\frac{\partial \theta_{n+1}^{(k)}}{\partial I_n^{(j)}}\right| \neq 0$$

Exemplo: Osciladores acoplados

$$\dot{\varphi}_k = \Omega_k + \varepsilon f(\varphi_{k-1} - \varphi_k) + \varepsilon f(\varphi_{k+1} - \varphi_k), \quad k = 1, \ldots, N,$$

N=4

$$\dot{\psi}_1 = \omega - 2\varepsilon \sin \psi_1 + \varepsilon \sin \psi_2,$$

$$\dot{\psi}_2 = 1 - 2\varepsilon \sin \psi_2 + \varepsilon \sin \psi_1 + \varepsilon \sin \psi_3,$$

$$\dot{\psi}_3 = \omega - 2\varepsilon \sin \psi_3 + \varepsilon \sin \psi_2,$$

4 osciladores de fase acoplados:

$$\dot{\psi}_1 = \omega - 2\varepsilon \sin \psi_1 + \varepsilon \sin \psi_2,$$
$$\dot{\psi}_2 = 1 - 2\varepsilon \sin \psi_2 + \varepsilon \sin \psi_1 + \varepsilon \sin \psi_3,$$
$$\dot{\psi}_3 = \omega - 2\varepsilon \sin \psi_3 + \varepsilon \sin \psi_2,$$

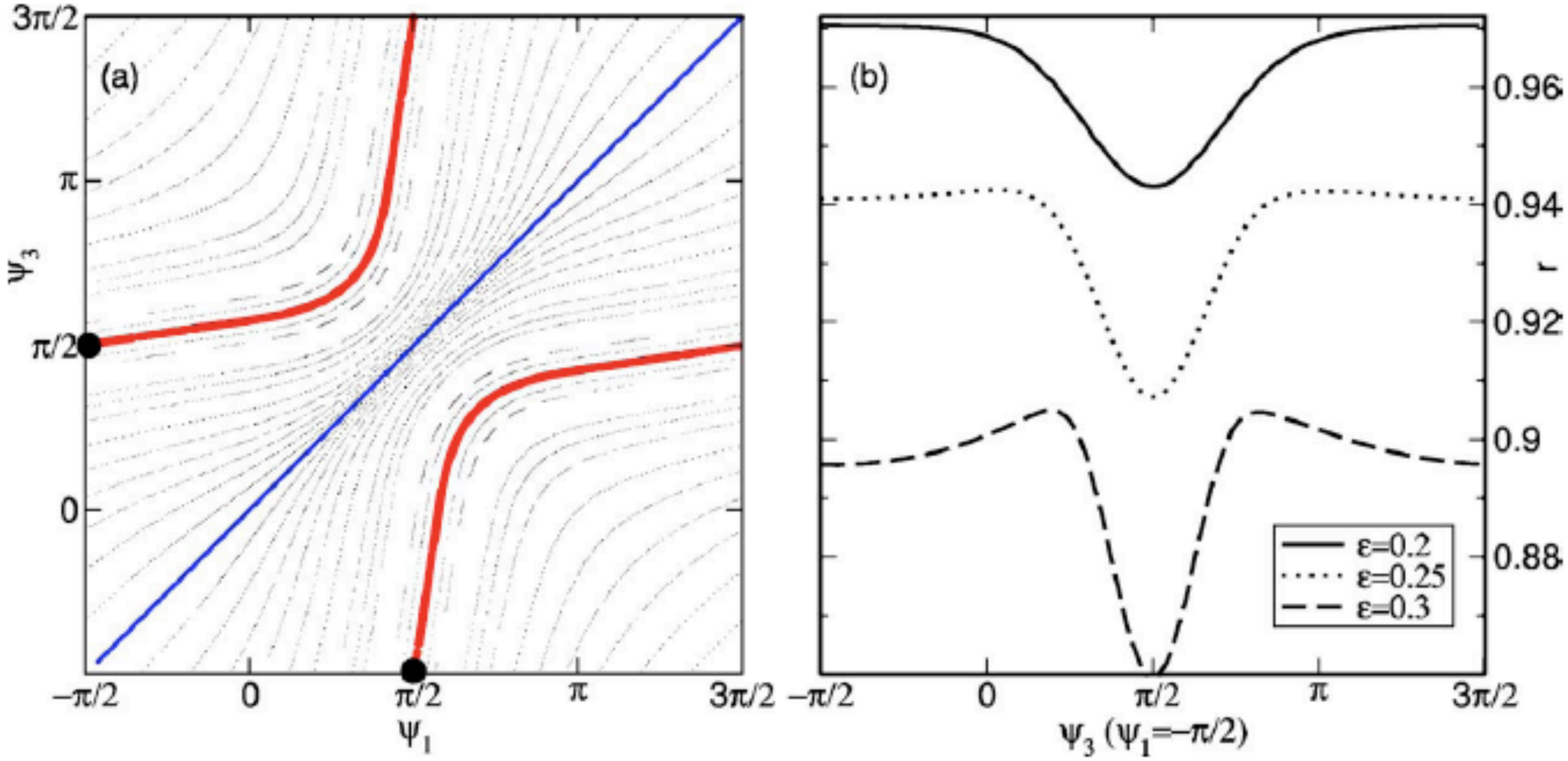
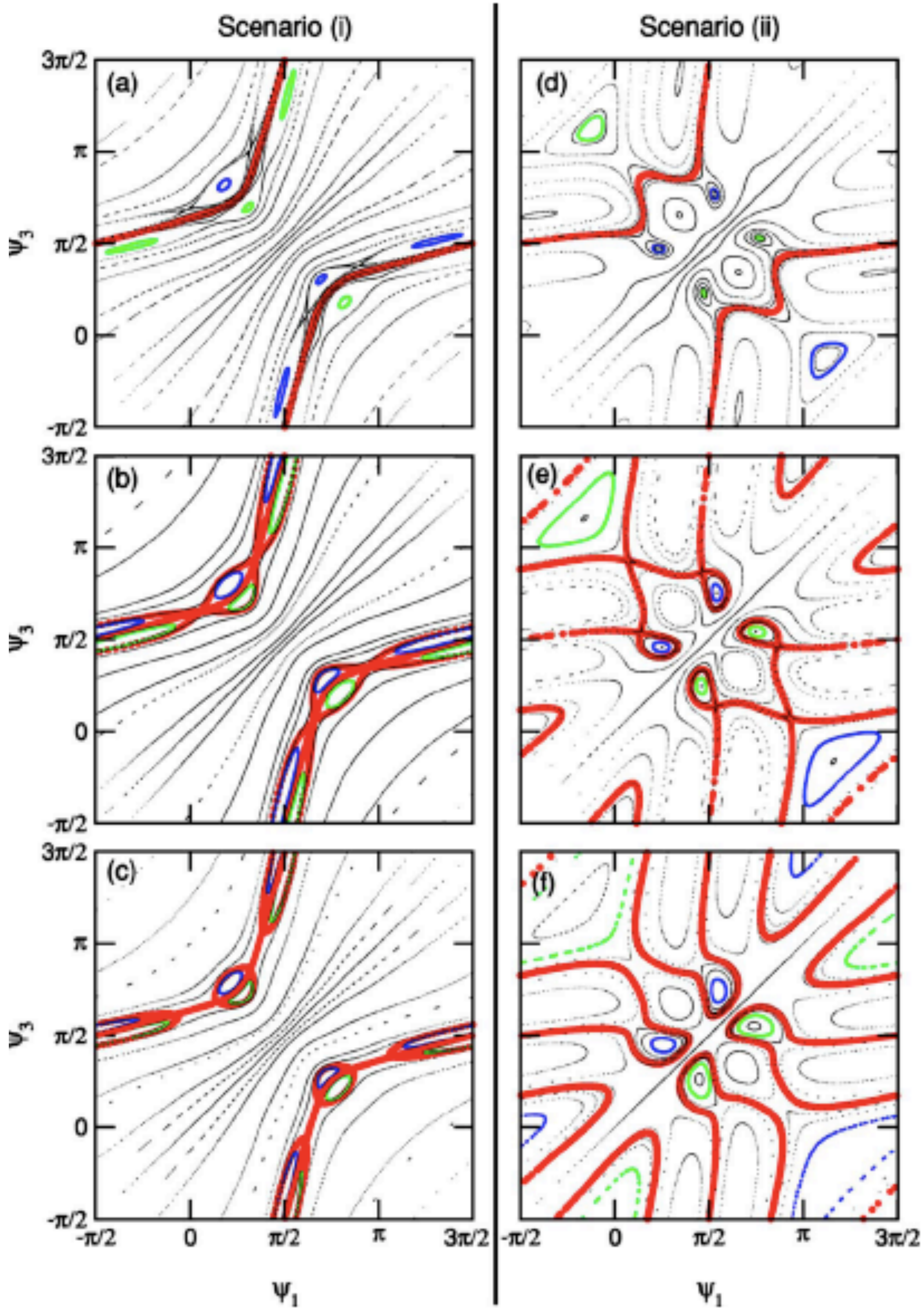


FIG. 1. (Color online) (a) Poincaré section ($\psi_2 = \pi/2$) of the system in Eq. (5) for $\omega = 1, \varepsilon = 0.2$ where two nontwist tori are emphasized. IPs are marked with the symbol ●. (b) Rotation number of the tori as a function of the coordinate ψ_3 at fixed $\psi_1 = -\pi/2$ for $\omega = 1$ and different values of ε (see legend).

Sistema não Hamiltoneano mas com dinâmica quasi-Hamiltoneana e torus "não torcidos"!



Reconexões típicas de sistemas não torcionais também ocorrem em sistemas não Hamiltoneanos

FIG. 2. (Color online) Poincaré section of system (5) for fixed $\varepsilon=0.25$ and different values of ω. Sequence (a) $\omega=0.868$, (b) $\omega=0.868\,760\,6$, and (c) $\omega=0.869$ illustrates the collision of 3:4 island chains. Sequence (d) $\omega=0.801$, (e) $\omega=0.801\,523$, and (f) $\omega=0.802$ illustrate a reconnection around 2:3 resonances.

Para o transporte de trajetórias é importante determinar quais parâmetros (ε,ω) o torus existe (divide o espaço de fases).

Método:

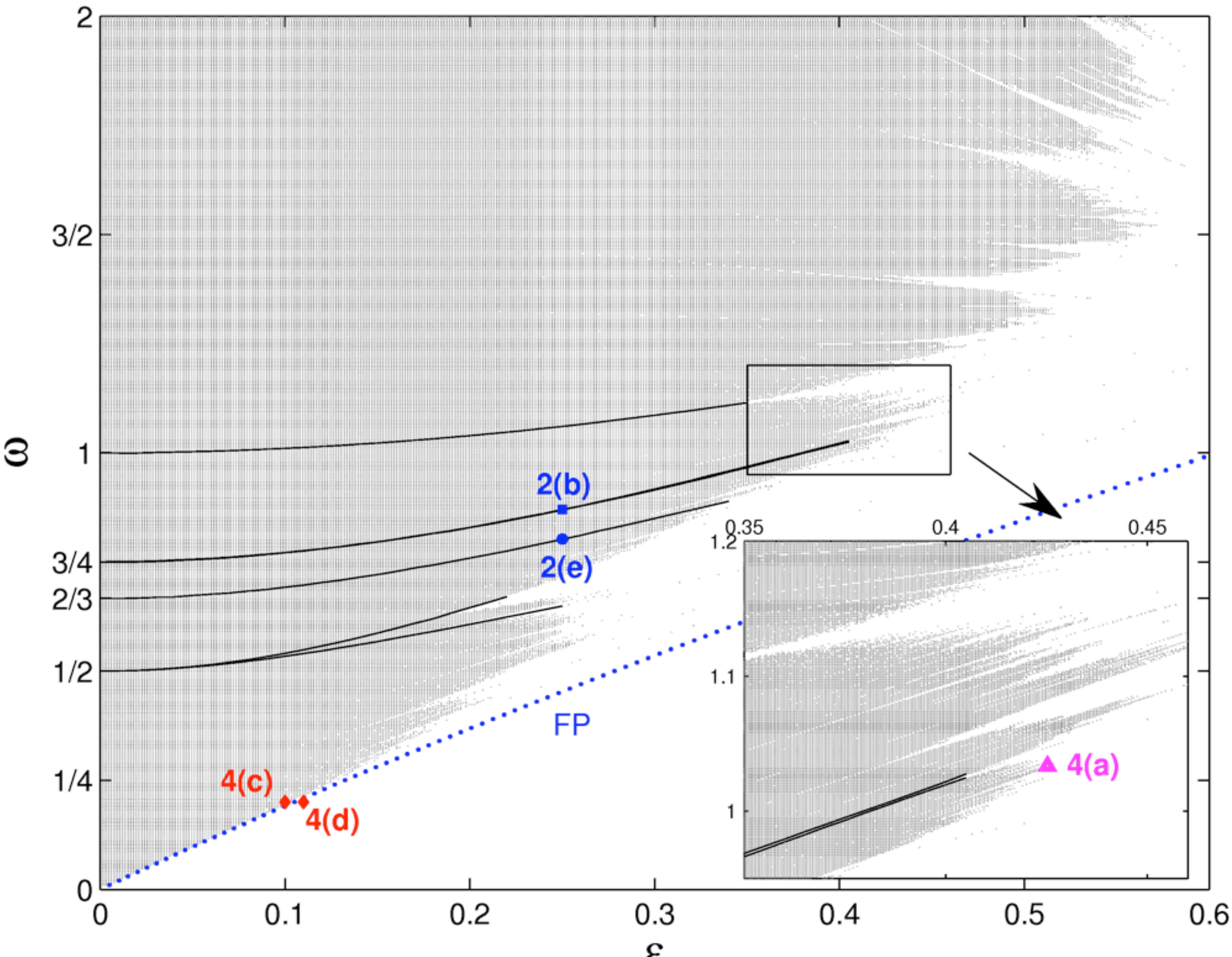
Verificar se uma trajetória que tem de pertencer ao torus (IP) satisfaz o teorema de Slater (e.g., máximo 3 Ts distintos).

Particularmente útil quando:

- Grande número de parâmetros (ε,ω) tem de ser varridos.
- Sistema de tempos contínuo (difícil integração/sessão de Poincaré)
- Parâmetros (ε,ω) próximos ao rompimento são escolhidos. Nesse caso ilhas ("stickiness") fazem demais métodos muito lentos

Limitação:

-N=4, i.e., mapas bi-dimensionais



2
3/2
1
3/4
2/3
1/2
1/4
0
ω
0
0.1
0.2
0.3
0.4
0.5
0.6
ε
2(b)
2(e)
FP
4(c)
4(d)
0.35
0.4
0.45
1.2
1.1
1
4(a)

Rompimento do torus

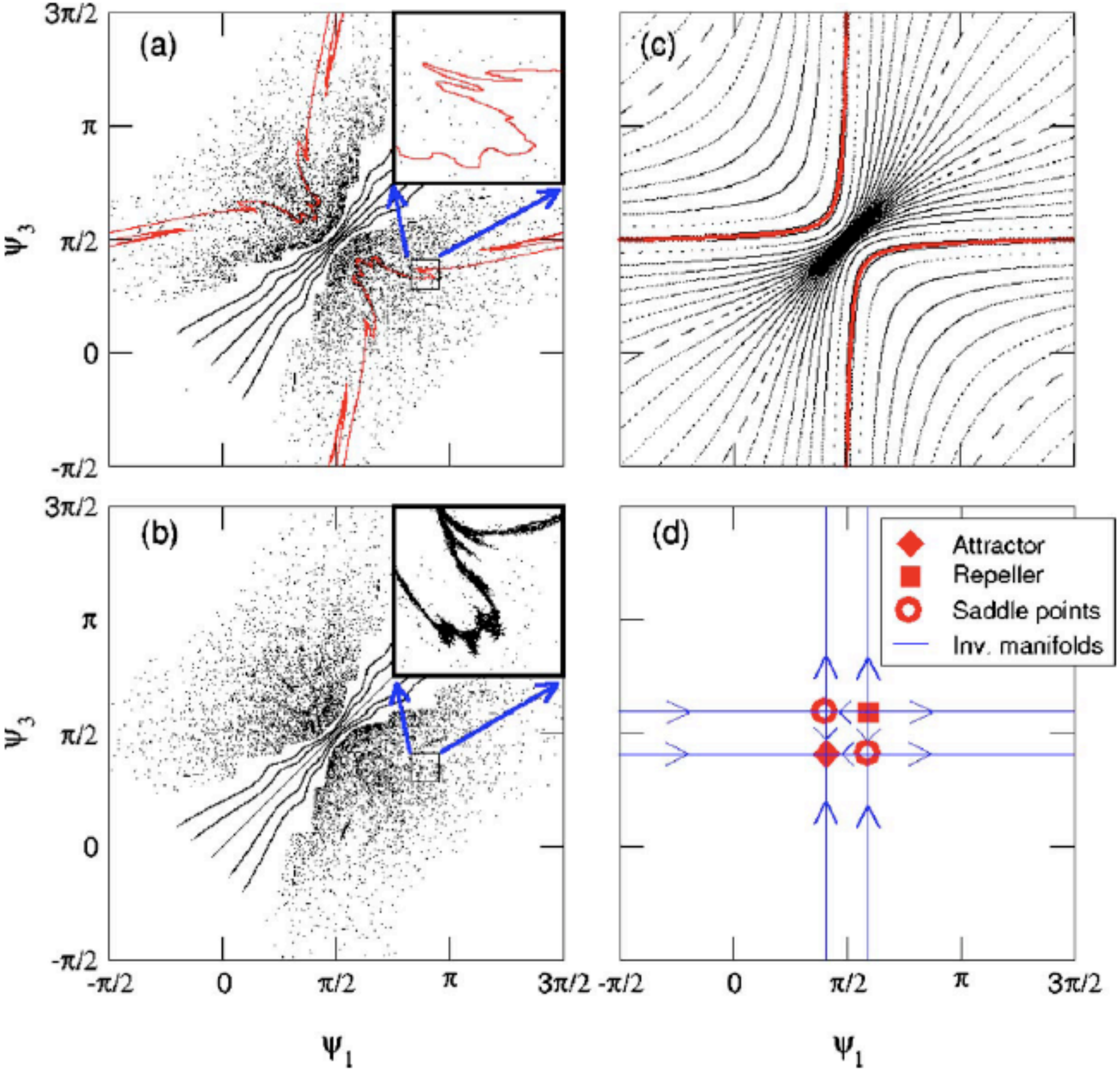


FIG. 4. (Color online) Two different routes for the breakup of the shearless torus: (a), (b) Hamiltonian-like through critical point and (c), (d) dissipative. (a) Torus near criticality $\varepsilon=0.425\,256$, $\omega=1.0335$ (the inset shows the fractal structure of the torus). (b) Torus after criticality $\varepsilon=0.425\,257$, $\omega=1.0335$ (the inset show that the torus is destroyed). (c) Near-integrable phase space $\varepsilon=0.1$, $\omega=0.2$. (d) Attracting fixed point $\varepsilon=0.11$, $\omega=0.2$.

Exemplo: mapa bi-dimensional

$$y_{n+1} = \frac{y_n + a\sin(2\pi x_n)}{1 + b y_n \sin(2\pi x_n)},$$

$$x_{n+1} = x_n + \cos(2\pi y_{n+1}) \quad \mathrm{mod}\ 1,$$

Jacobiano:

$$J = \frac{1 - ab\sin^2(2\pi x)}{[1 + by\sin(2\pi x)]^2}$$

Simetrias:

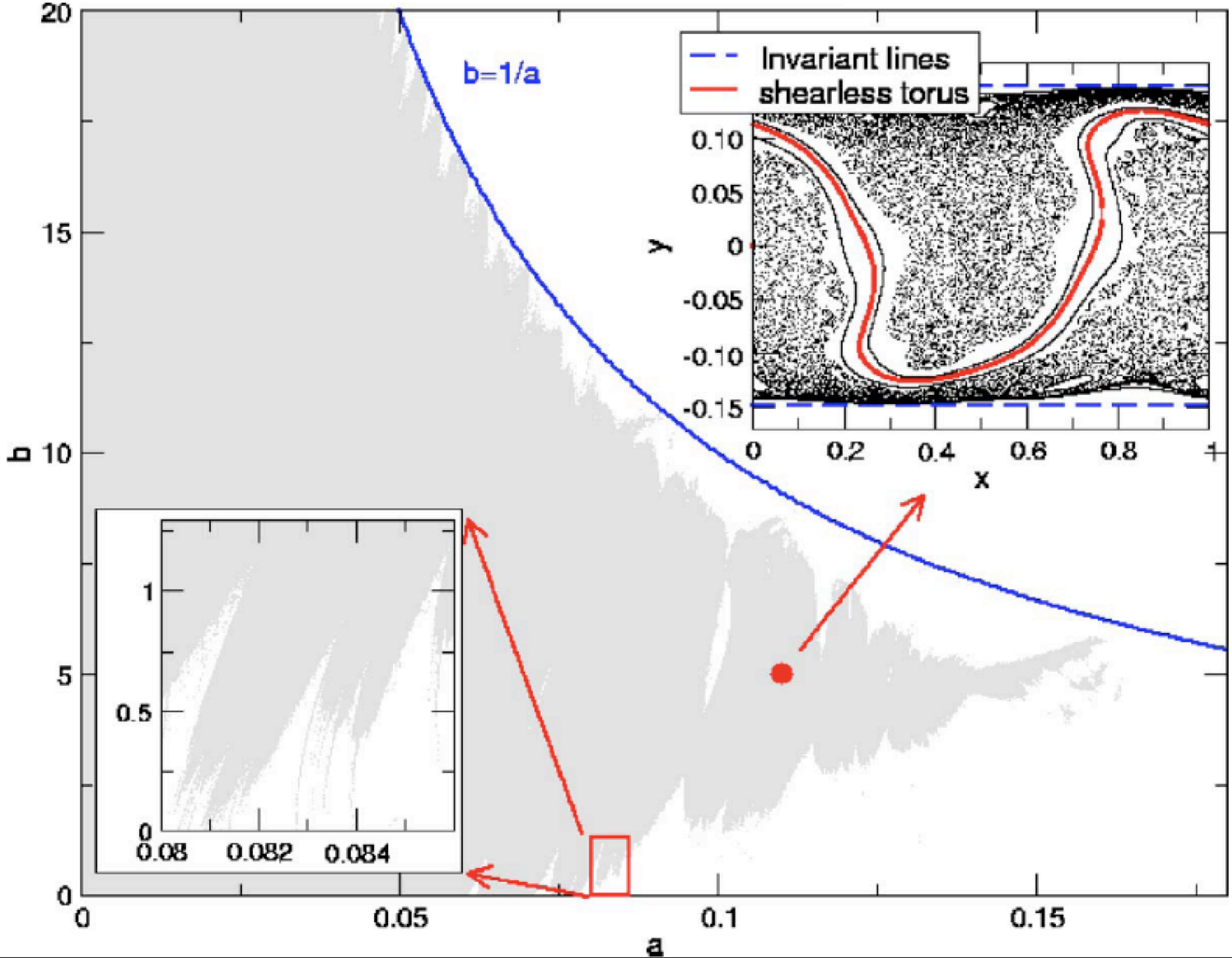
$$M_1: \quad x' = -x, \quad y' = \frac{y + a\sin(2\pi x)}{1 + by\sin(2\pi x)},$$

$$M_2: \quad x' = -x + \cos(2\pi y), \quad y' = y.$$

$$y_{n+1} = \frac{y_n + a\sin(2\pi x_n)}{1 + b y_n \sin(2\pi x_n)},$$

Exemplo: mapa bi-dimensional

$$x_{n+1} = x_n + \cos(2\pi y_{n+1}) \quad \mathrm{mod}\ 1,$$



Apresentação III:

mapa linear por partes
espaço de fases hierárquico

$$\begin{aligned} y_{n+1} &= y_n + Kf(x_n) \qquad \mod 1, \\ x_{n+1} &= x_n + y_{n+1} \qquad \mod 1, \end{aligned}$$

$$f(x_n) = \begin{cases} -x_n & \text{if } \quad 0 \le x_n < 1/4, \\ -1/2 + x_n & \text{if } 1/4 \le x_n < 3/4, \\ 1 - x_n & \text{if } 3/4 \le x_n \le 1, \end{cases}$$

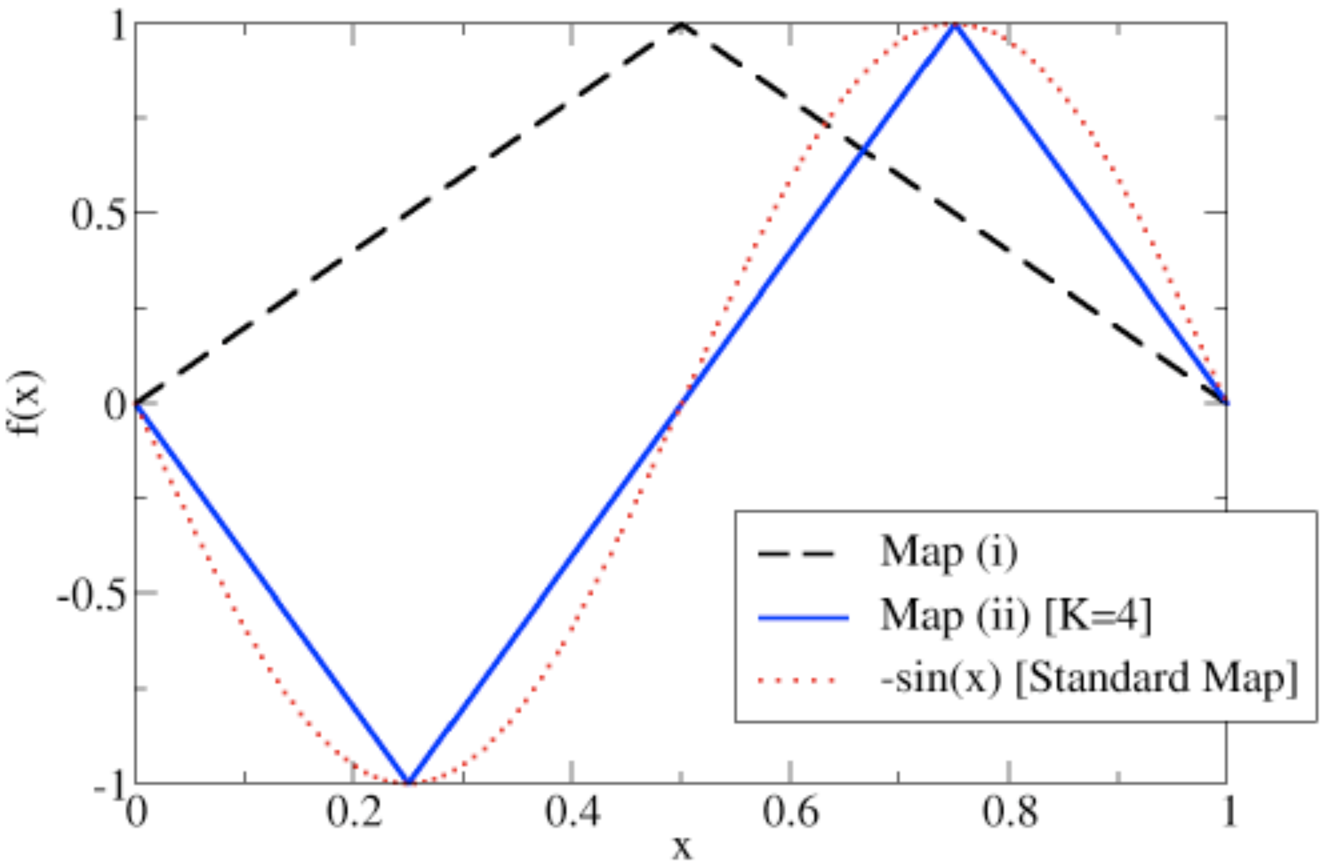
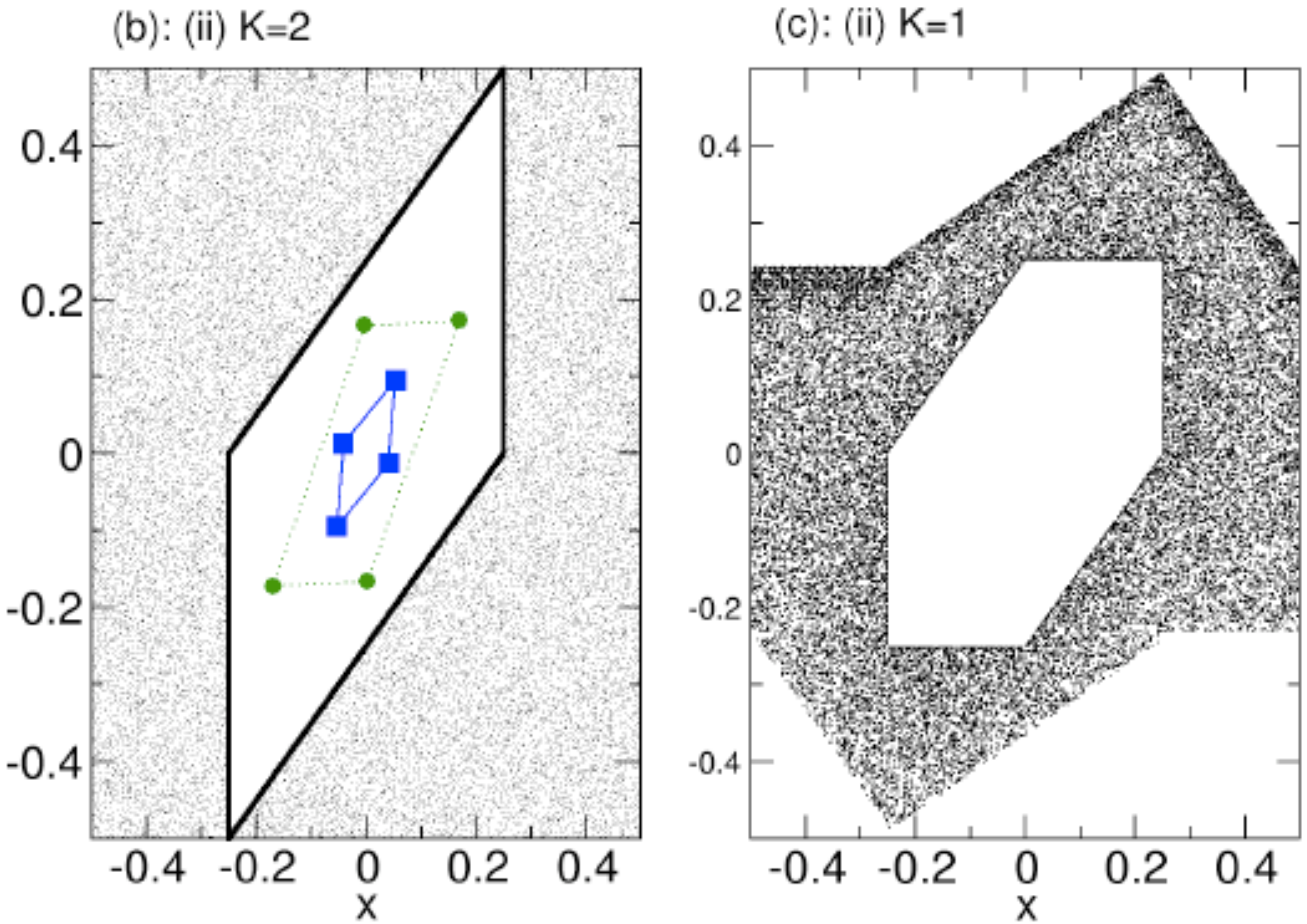
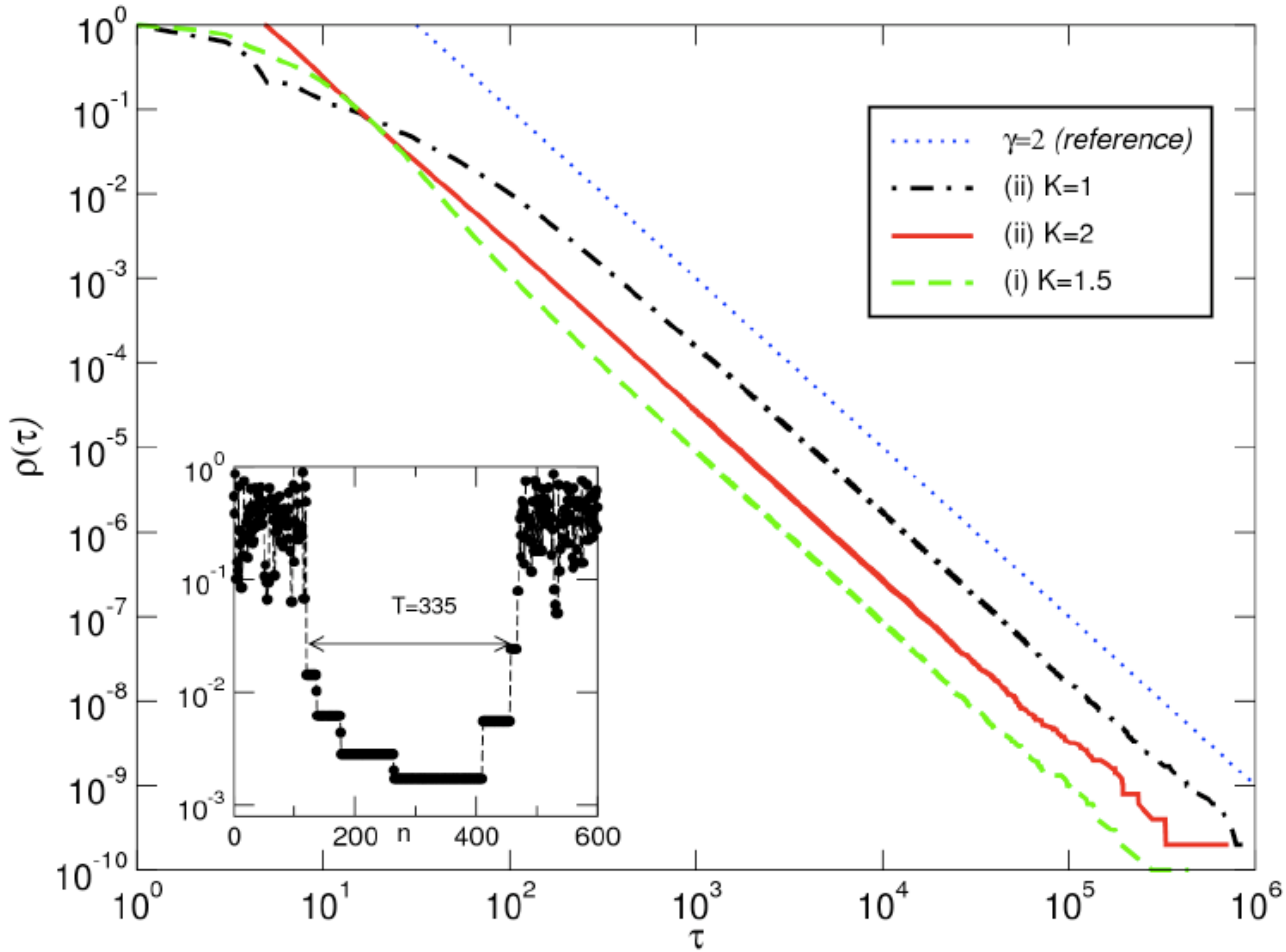


Figure 5.1: Illustration of the piecewise-linear functions (5.2) [map (i)] and (5.3) [map (ii)]. In the last case, the function was multiplied by a factor $K = 4$.

$$y_{n+1} = y_n + Kf(x_n) \mod 1,$$
$$x_{n+1} = x_n + y_{n+1} \mod 1,$$

$$f(x_n) = \begin{cases} -x_n & \text{if } \quad 0 \le x_n < 1/4, \\ -1/2 + x_n & \text{if } 1/4 \le x_n < 3/4, \\ 1 - x_n & \text{if } 3/4 \le x_n \le 1, \end{cases}$$

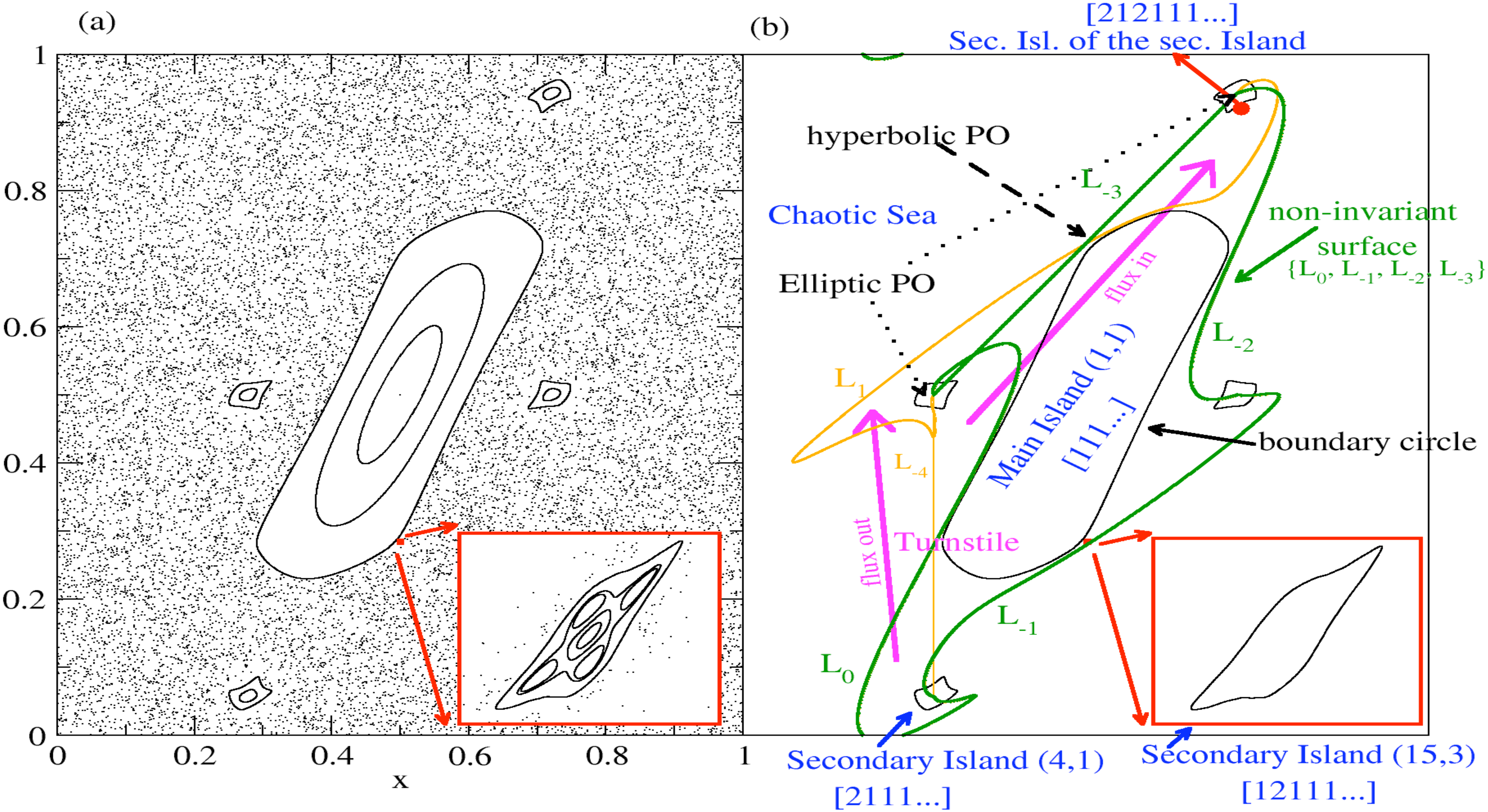


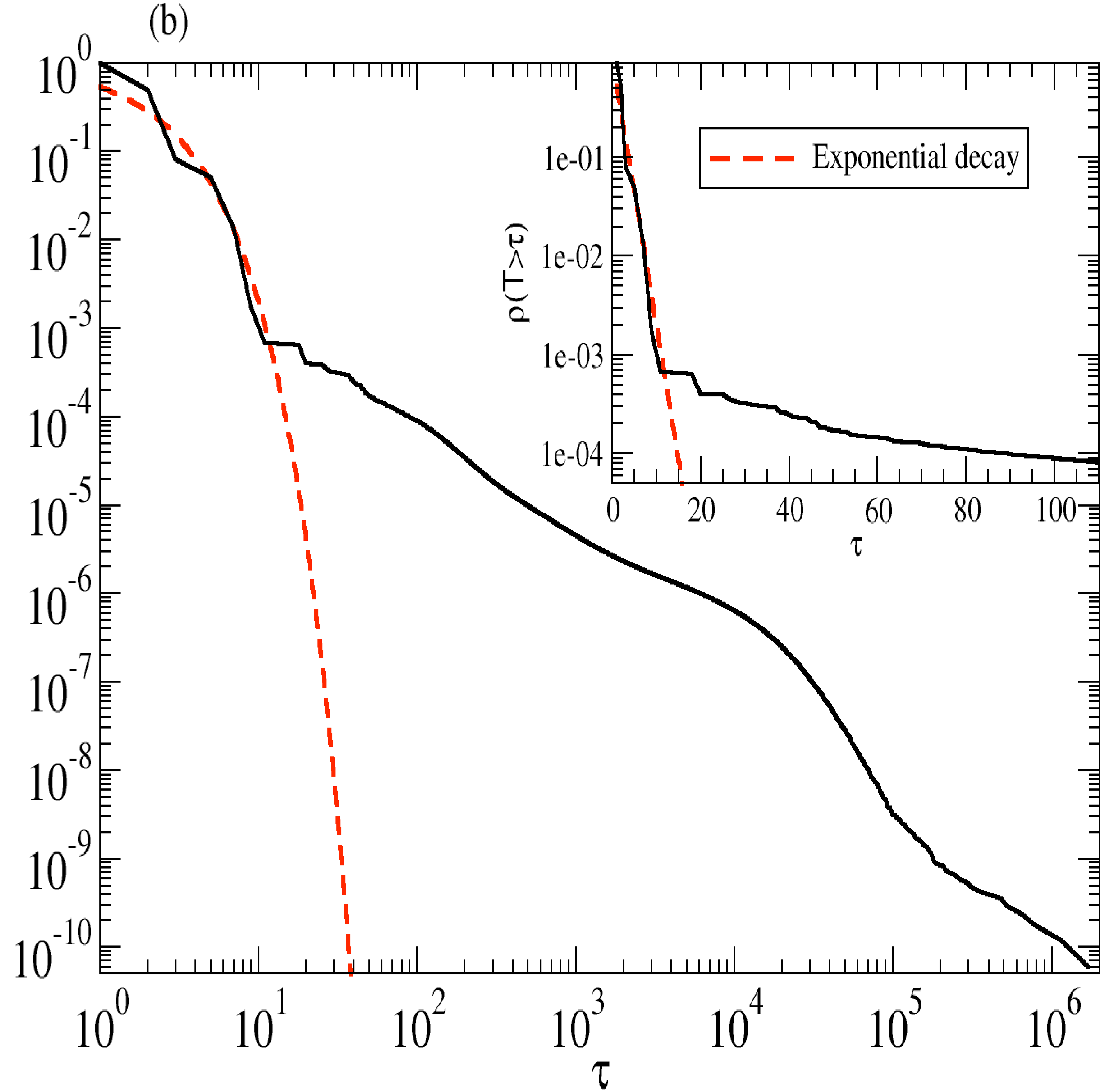


ρ(τ)
τ
γ=2 (reference)
(ii) K=1
(ii) K=2
(i) K=1.5
T=335
n

Caso genérico / hierárquico

$$K = 0.45$$





(b)
ρ(T>τ)
τ
Exponential decay

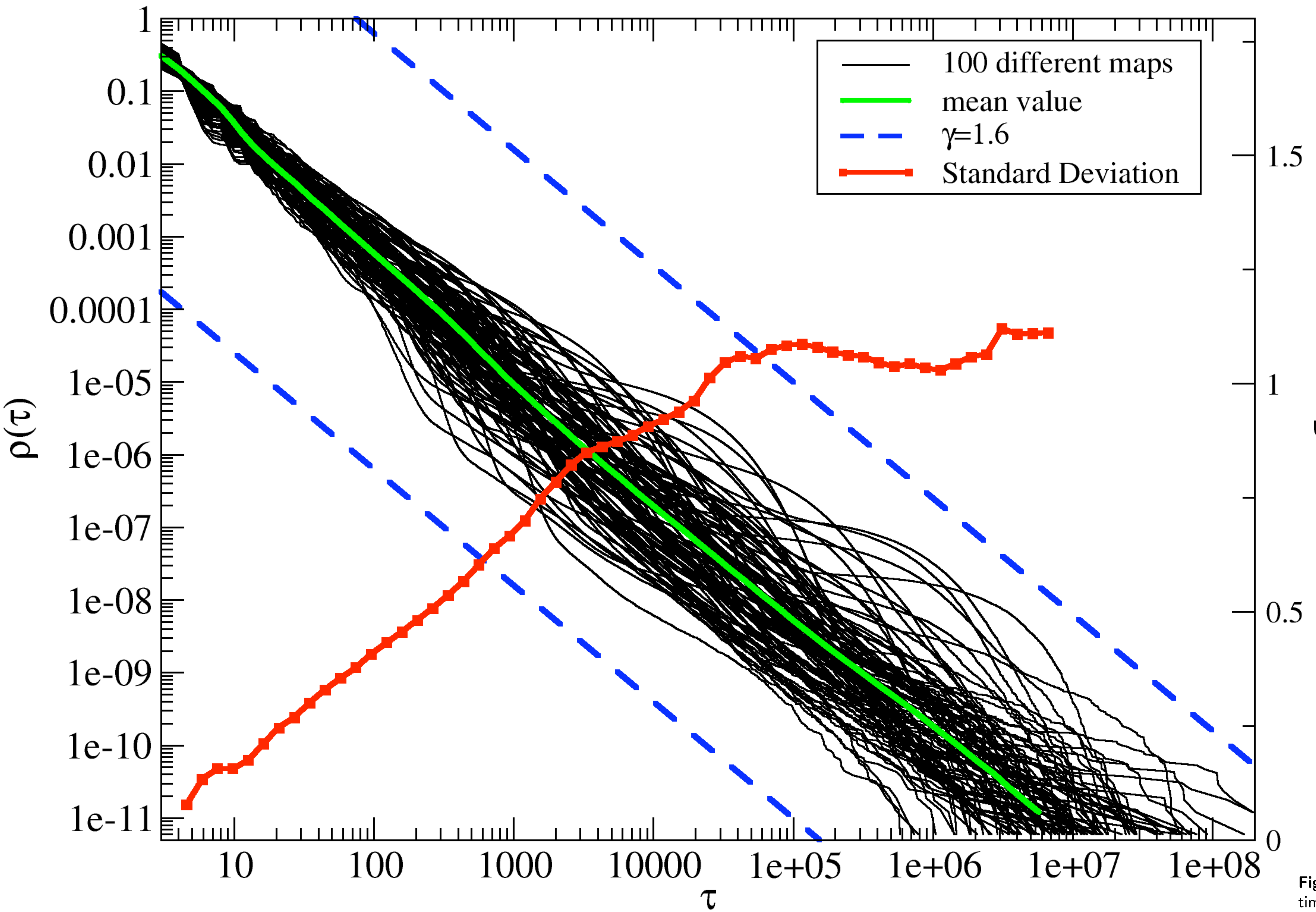


Figure 2.3: (Color online) Sticking time distribution $\rho(\tau)$ for 100 different standard maps (2.14) with a constant $K^\dagger$ added to the y equation: $K \in [0.5, 0.6], K^\dagger \in [0, 0.2]$. The central green (gray) curve is the average [for fixed $\rho(\tau)$] over all curves, and the red curve (axis on the right) corresponds to the standard deviation of the curves (for fixed $\rho(\tau)$ projected to the x-axis). The further parameters are equivalent to those of Fig. 6.1b below.

Apresentação IV:

Efeito de ruído branco e altas dimensões no aprisionamento de trajetórias

Coupled standard maps:

→ 2.1 Motivation / model

2.2 Noise perturbation

2.3 High dimensional

Qual o problema?
(do ponto de vista de Mec. Estatística)

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Violate the hypothesis of strong chaos:

1. Ergodicity, i.e., negligible measure of regular components

2. Strong mixing, i.e., fast decay of correlations

Qual o problema?
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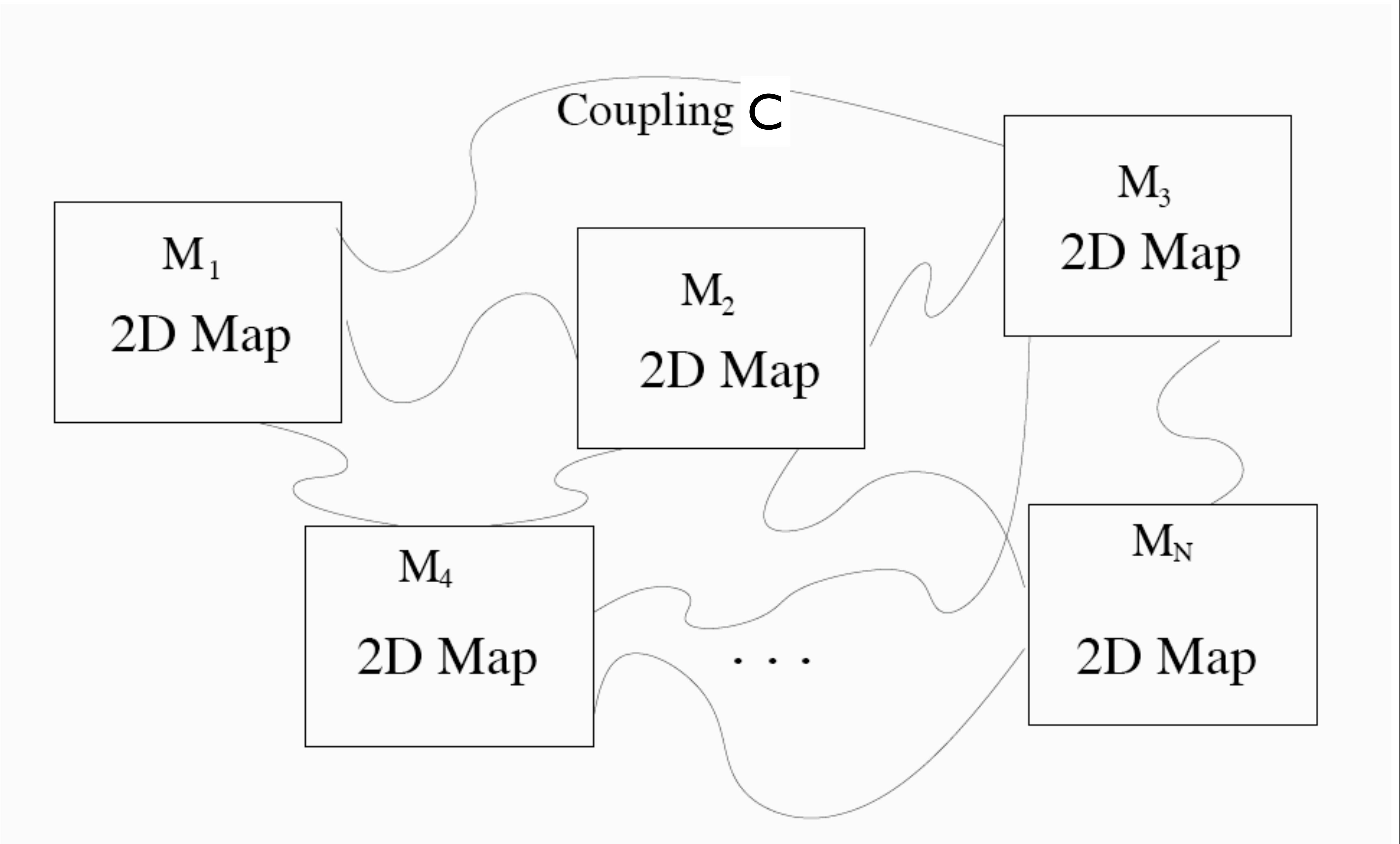
Violate the hypothesis of strong chaos:

1. Ergodicity, i.e., negligible measure of regular components

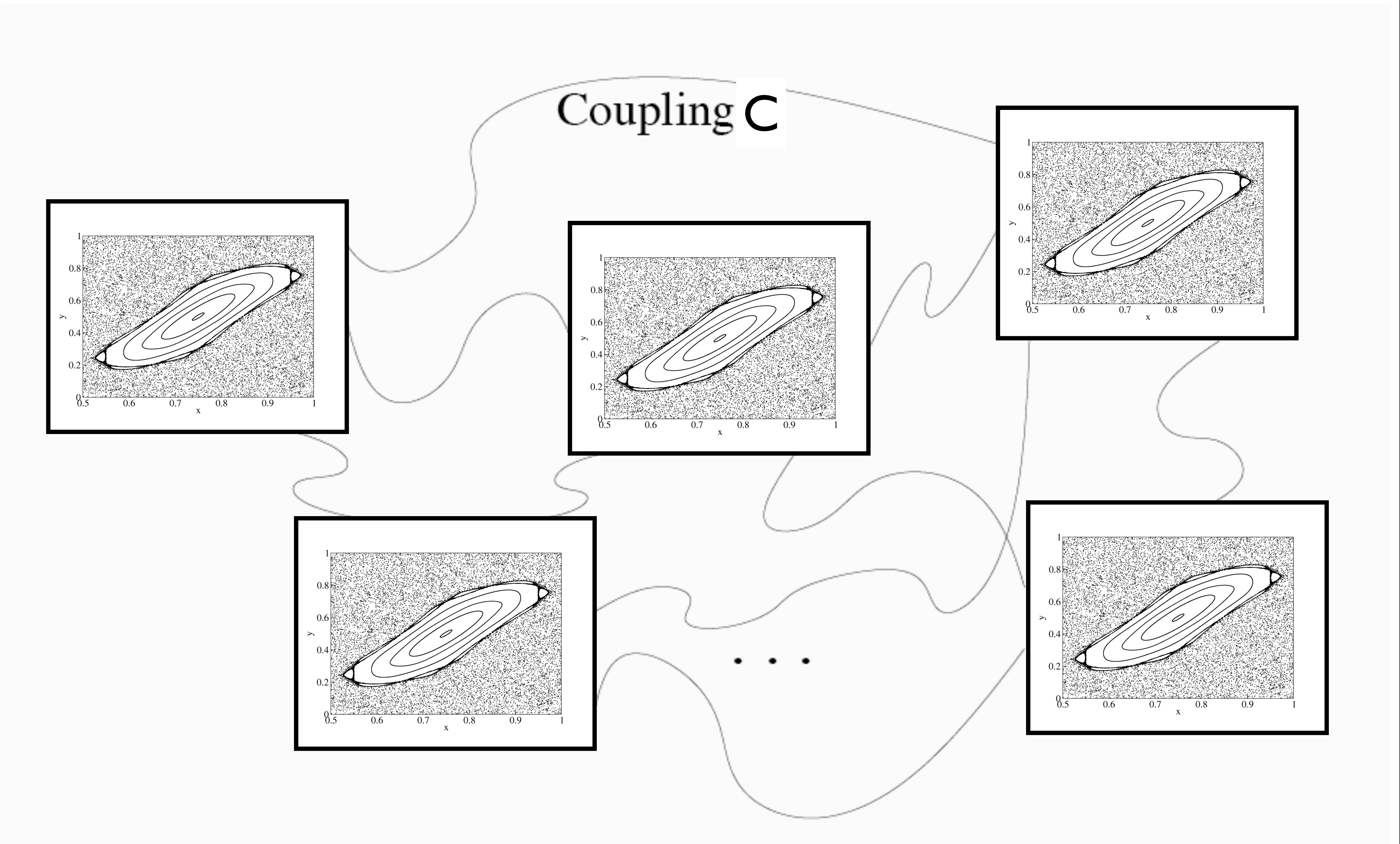
2. Strong mixing, i.e., fast decay of correlations

What happens for increasing phase space dimension?

Coupled symplectic maps model



Coupled symplectic maps model



Coupled symplectic maps model

Map $(\boldsymbol{p}', \boldsymbol{q}') = \boldsymbol{M}_N(\boldsymbol{p}, \boldsymbol{q})$ is symplectic iff: $\boldsymbol{S}_N = \left(\frac{\partial \boldsymbol{M}_N}{\partial \boldsymbol{x}}\right)^\dagger \boldsymbol{S}_N \left(\frac{\partial \boldsymbol{M}_N}{\partial \boldsymbol{x}}\right)$

$$\boldsymbol{x} = (q_1, ..., q_N, p_1, ..., p_N) \qquad \boldsymbol{S}_N = \begin{pmatrix} \boldsymbol{0}_N & -\boldsymbol{I}_N \\ \boldsymbol{I}_N & \boldsymbol{0}_N \end{pmatrix}$$

Coupled symplectic maps model

Map $(\boldsymbol{p}', \boldsymbol{q}') = \boldsymbol{M}_N(\boldsymbol{p}, \boldsymbol{q})$ is symplectic iff: $\boldsymbol{S}_N = \left(\frac{\partial \boldsymbol{M}_N}{\partial \boldsymbol{x}}\right)^\dagger \boldsymbol{S}_N \left(\frac{\partial \boldsymbol{M}_N}{\partial \boldsymbol{x}}\right)$

$\boldsymbol{x} = (q_1, ..., q_N, p_1, ..., p_N)$ $\quad \boldsymbol{S}_N = \begin{pmatrix} \boldsymbol{0}_N & -\boldsymbol{I}_N \\ \boldsymbol{I}_N & \boldsymbol{0}_N \end{pmatrix}$

$$\mathfrak{M} = \boldsymbol{C} \circ \boldsymbol{M}$$

The maps $\boldsymbol{M} = (M_1, \ldots, M_N)$ and couplings $\boldsymbol{C} = (C_1, \ldots, C_N)$ are given as

$$M_i \begin{pmatrix} p_i \\ q_i \end{pmatrix} = \begin{pmatrix} p_i + K_i \sin(2\pi q_i) & \mod 1 \\ q_i + p_i + K_i \sin(2\pi q_i) & \mod 1 \end{pmatrix}$$

$$C_i \begin{pmatrix} p_i \\ q_i \end{pmatrix} = \begin{pmatrix} p_i + \sum_{j=1}^{N} \xi_{i,j} \sin[2\pi(q_i - q_j)] \\ q_i \end{pmatrix}$$

$\boldsymbol{C}$ is symplectic iff $\xi_{i,j} = \xi_{j,i}$. We use all-to-all coupling with $\xi_{i,j} = \frac{\xi}{\sqrt{N-1}}$

For large N, weak coupled chaotic maps q_i, q_j are almost uncorrelated:

$$\Delta p_i = \frac{\xi}{\sqrt{N-1}} \sum_{j=1}^{N} \sin[2\pi(q_i - q_j)] \approx \xi\delta$$

Coupled standard maps:

2.1 Motivation / model

→ 2.2 Noise perturbation

2.3 High dimensional

2.2 Noise perturbation

$$y_{i+1} = y_i + K \sin(2\pi x_i) \quad \text{mod } 1,$$
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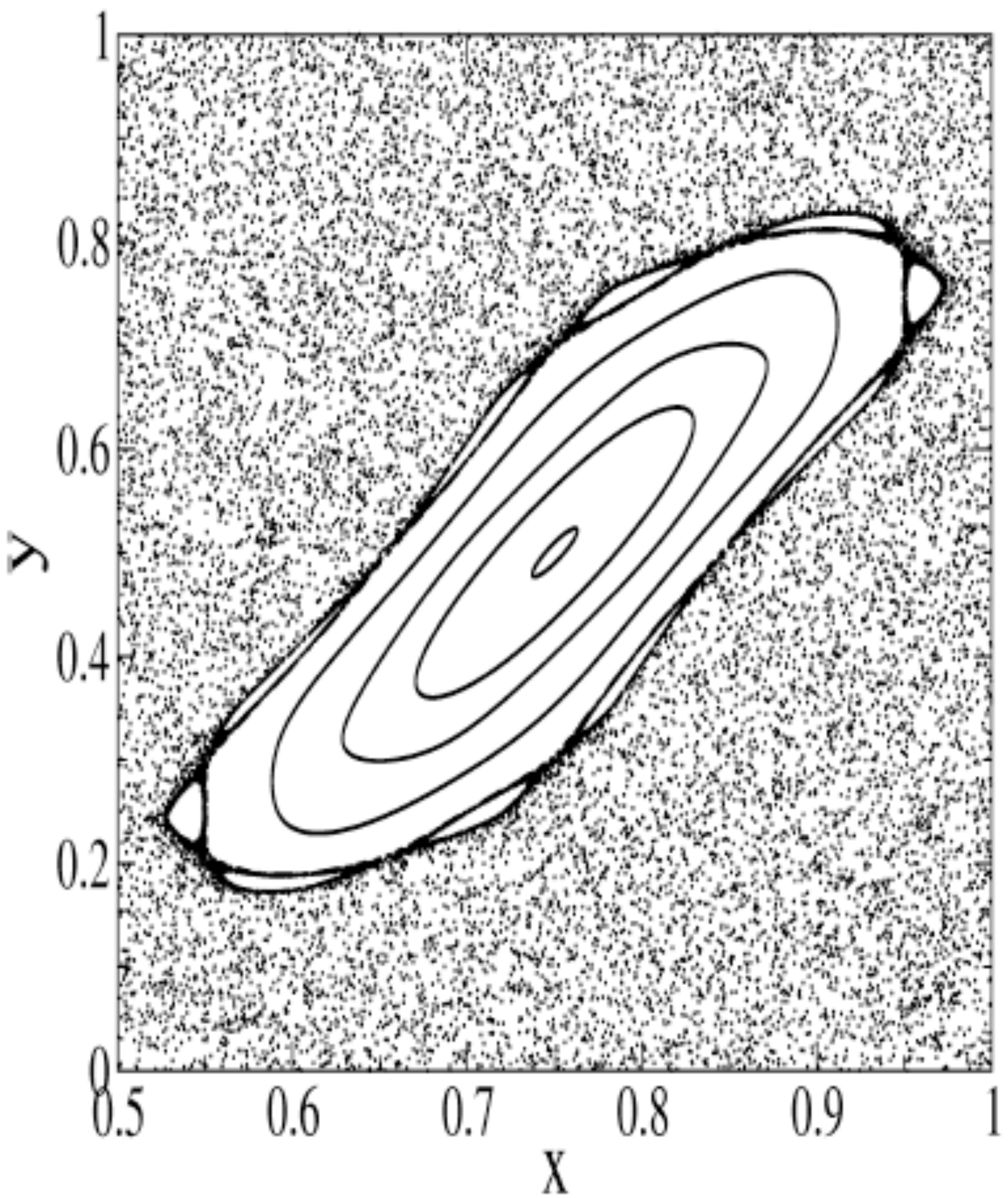
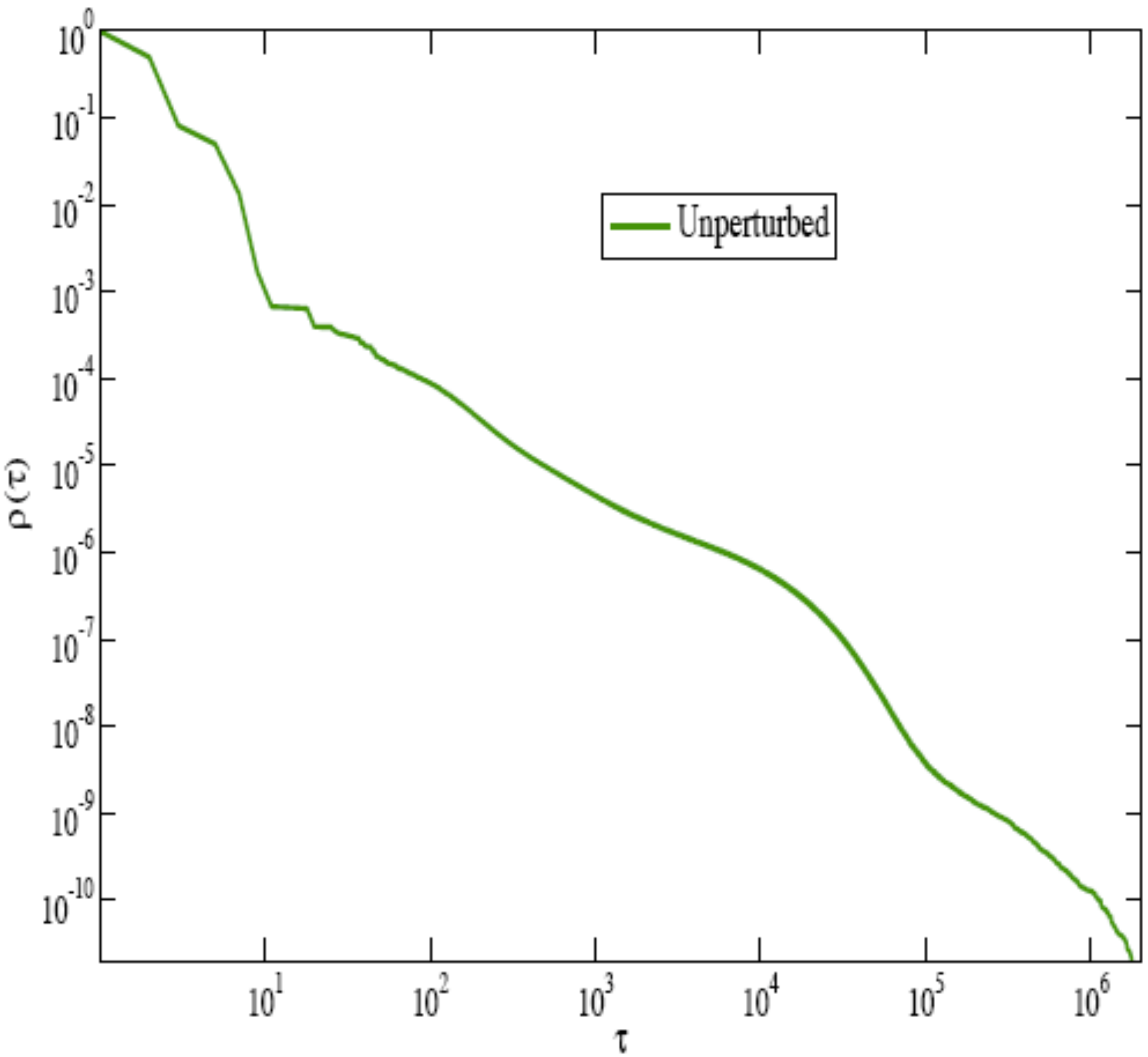
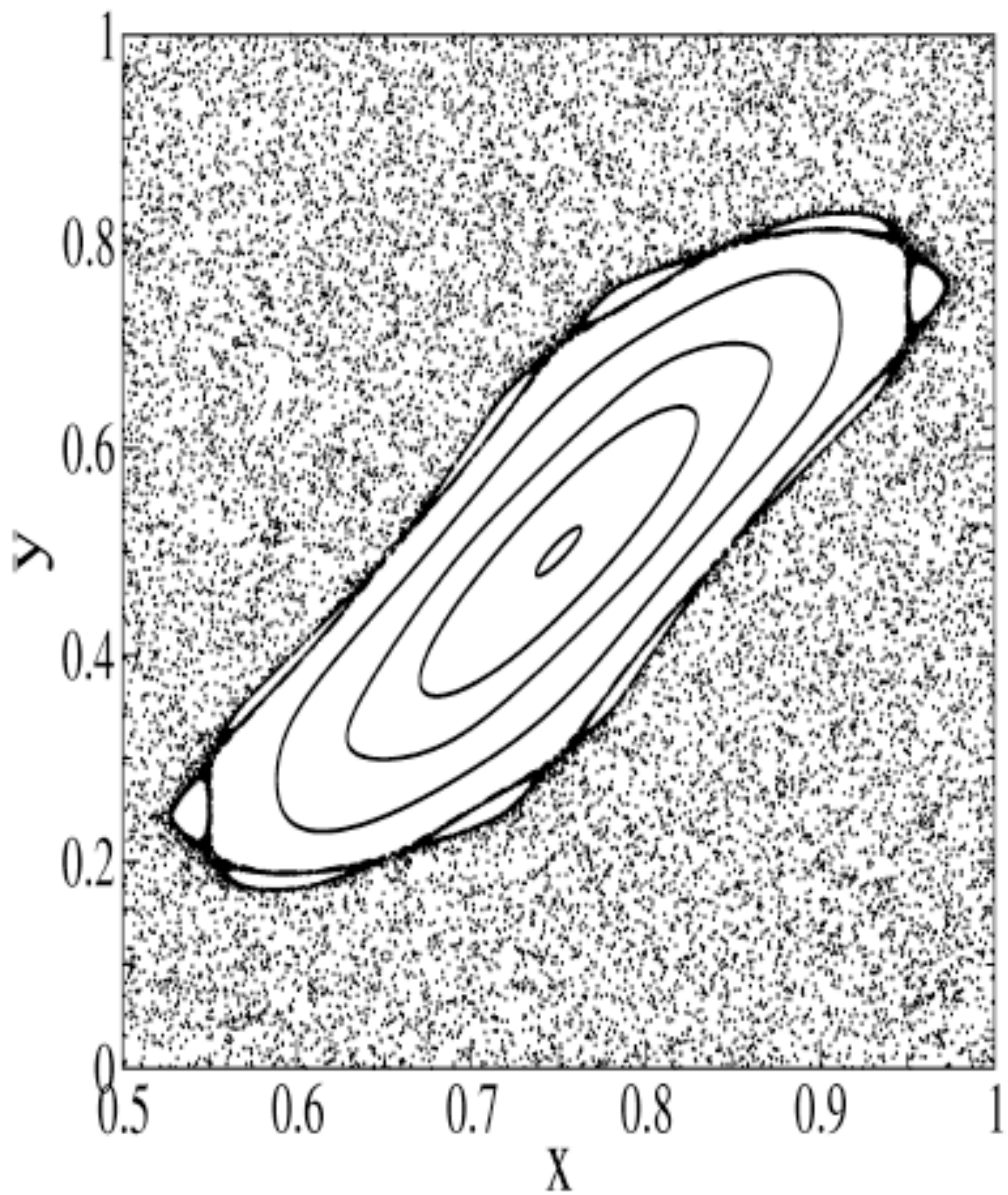
In the following K=0.52.

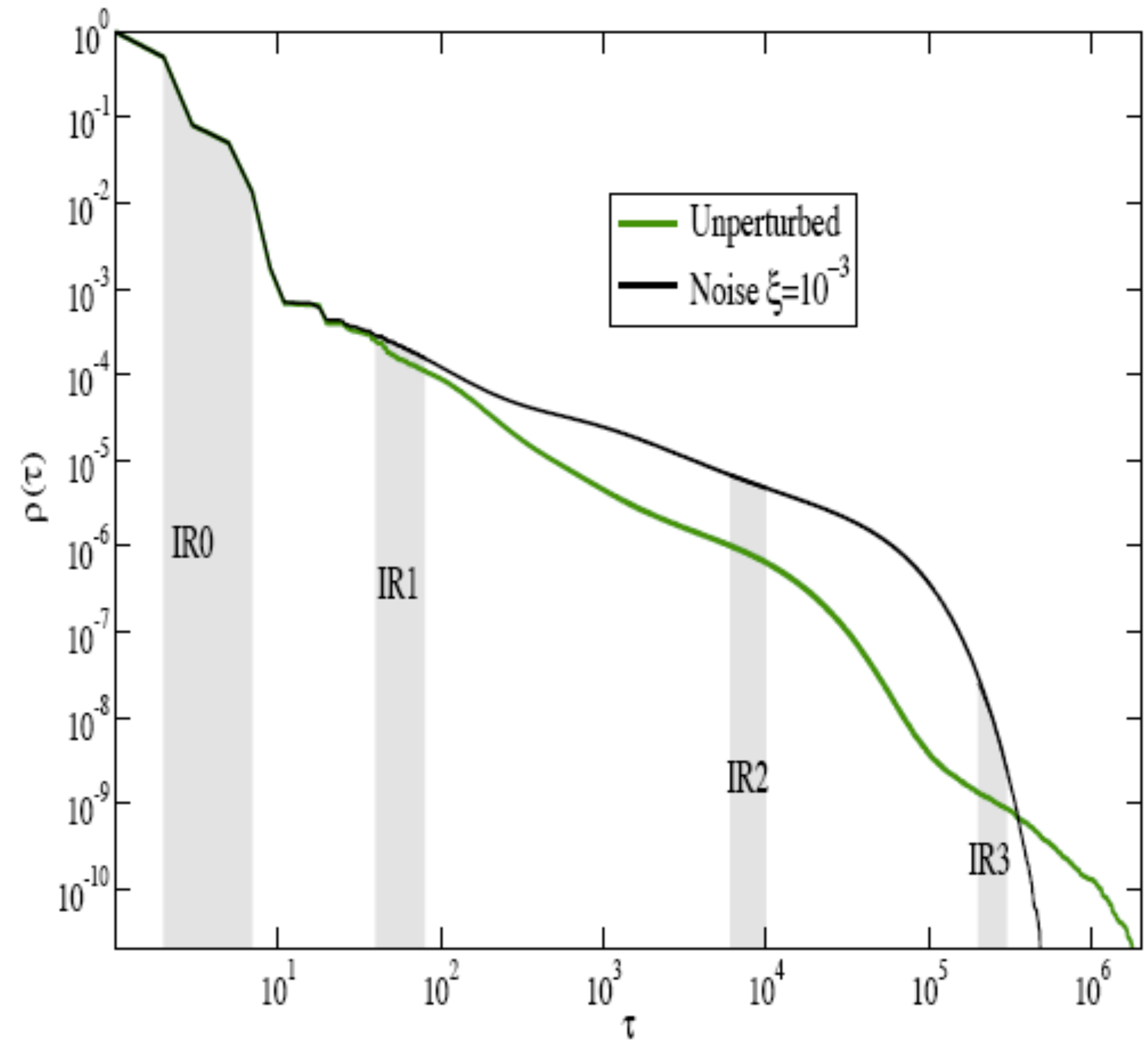




2.2 Noise perturbation

$$y_{i+1} = y_i + K \sin(2\pi x_i) \quad \text{mod } 1,$$
$$x_{i+1} = x_i + y_{i+1} + \xi\delta_i \quad \text{mod } 1,$$





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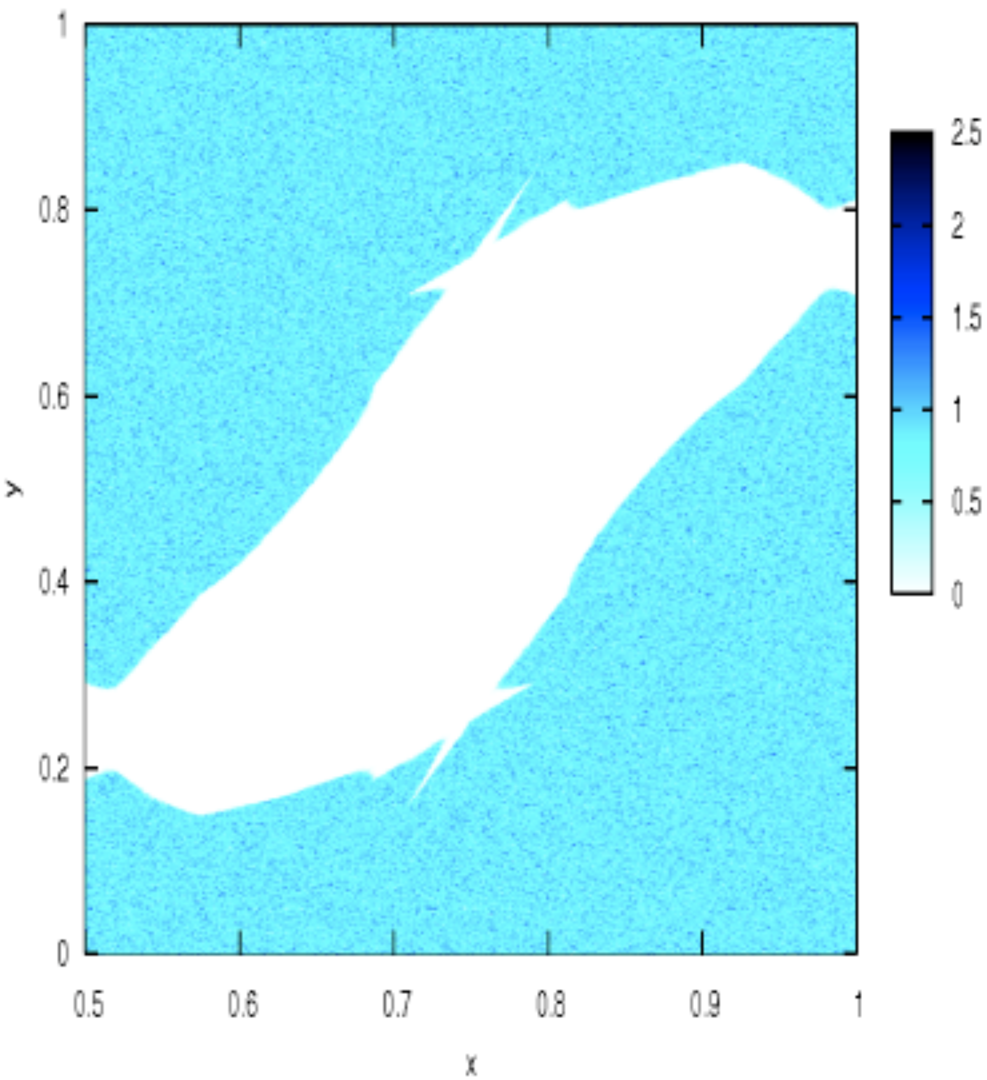

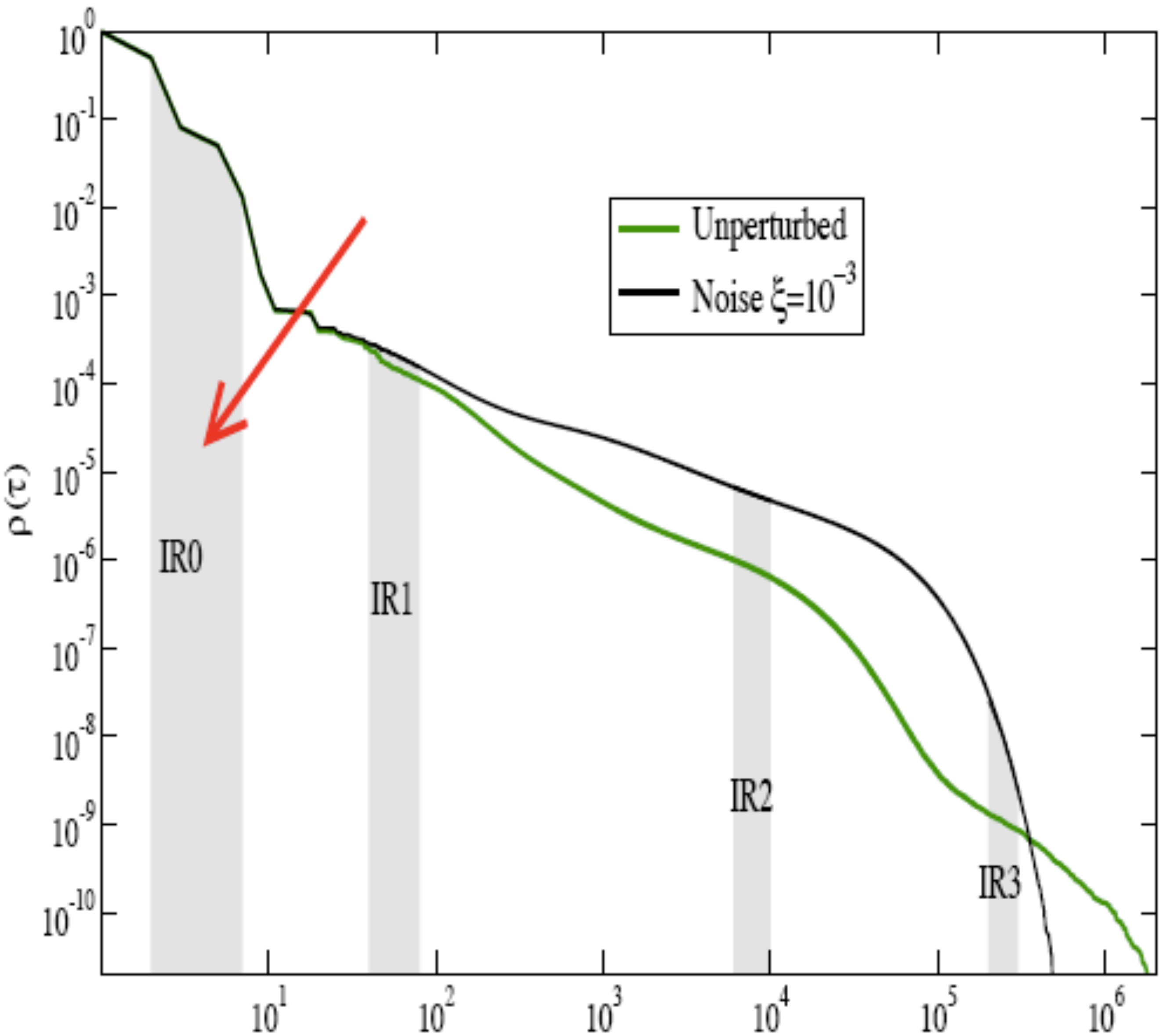

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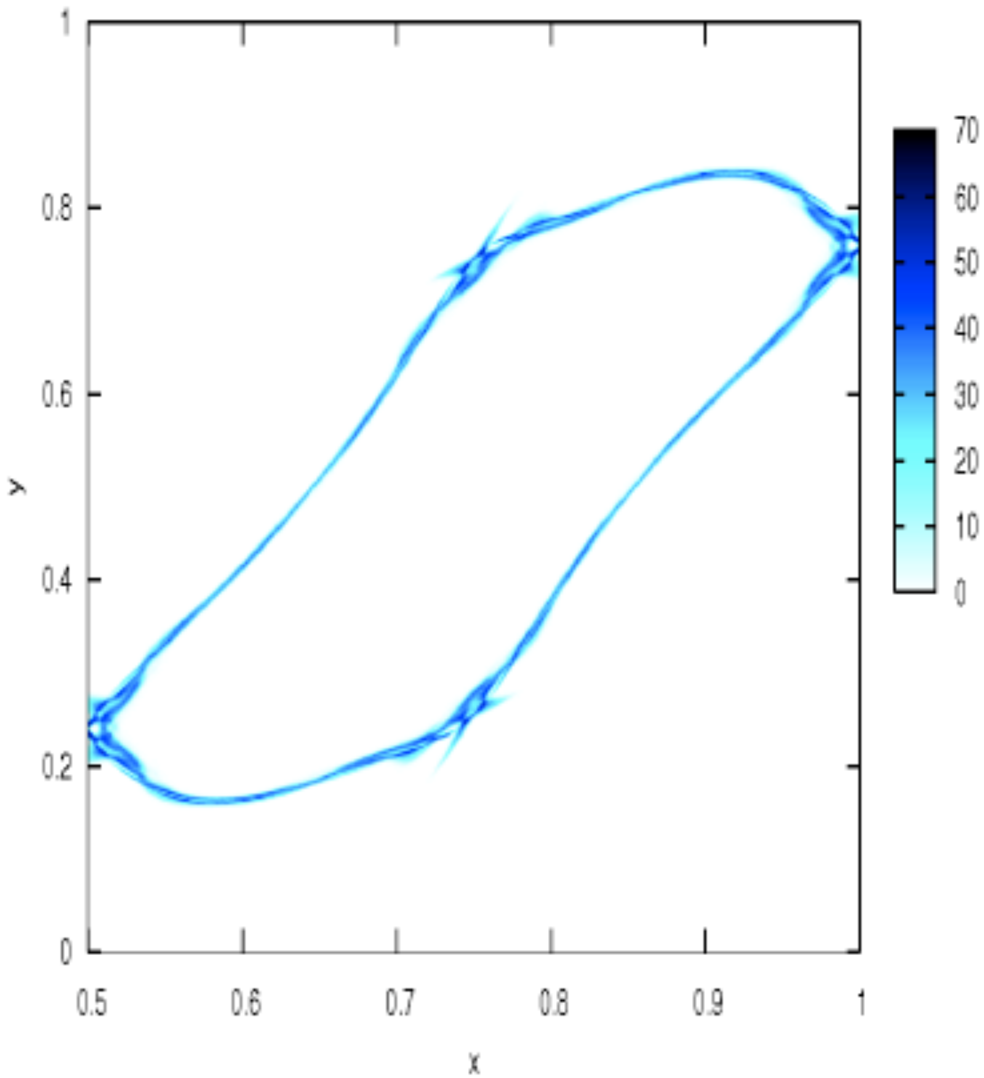

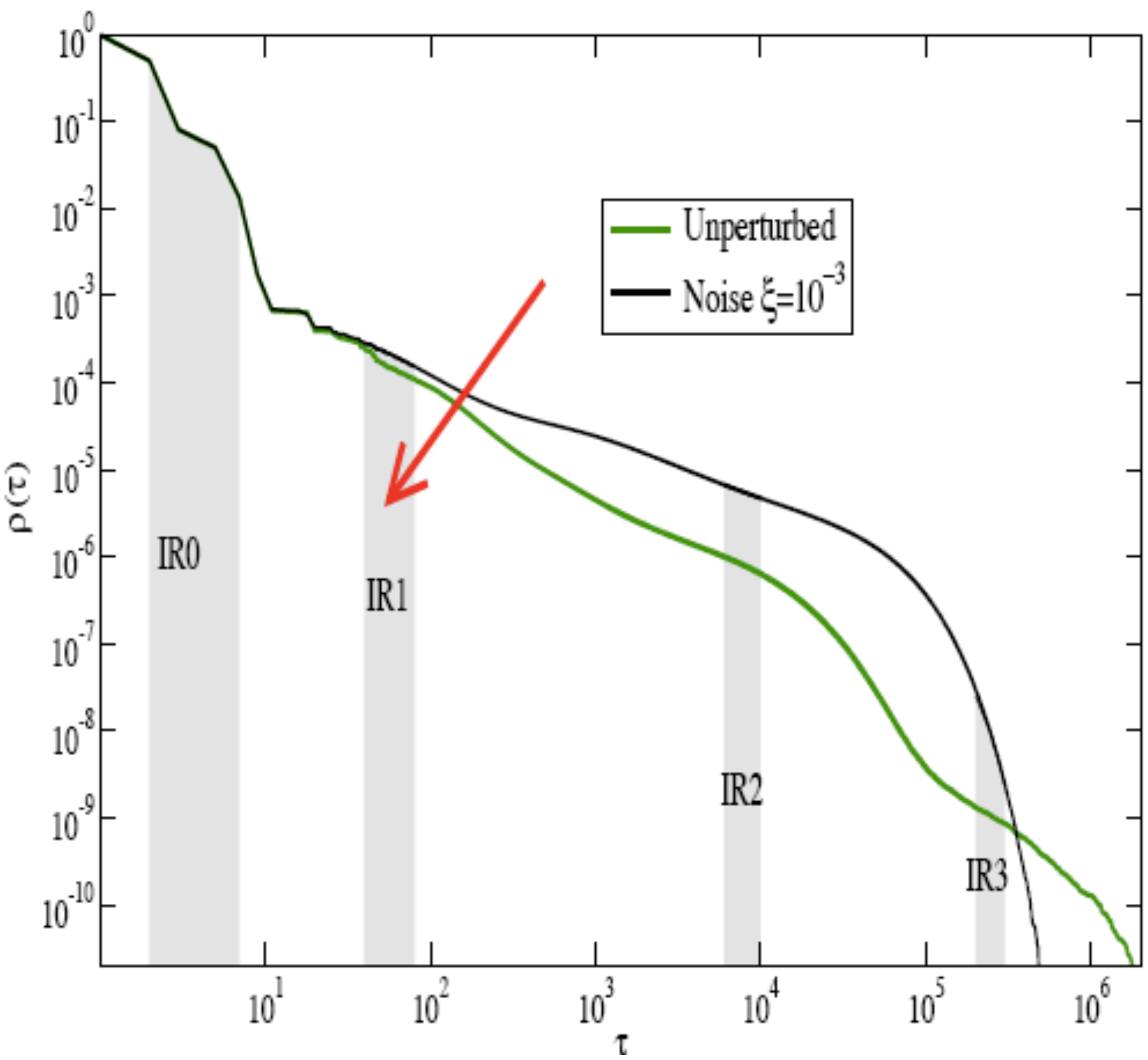

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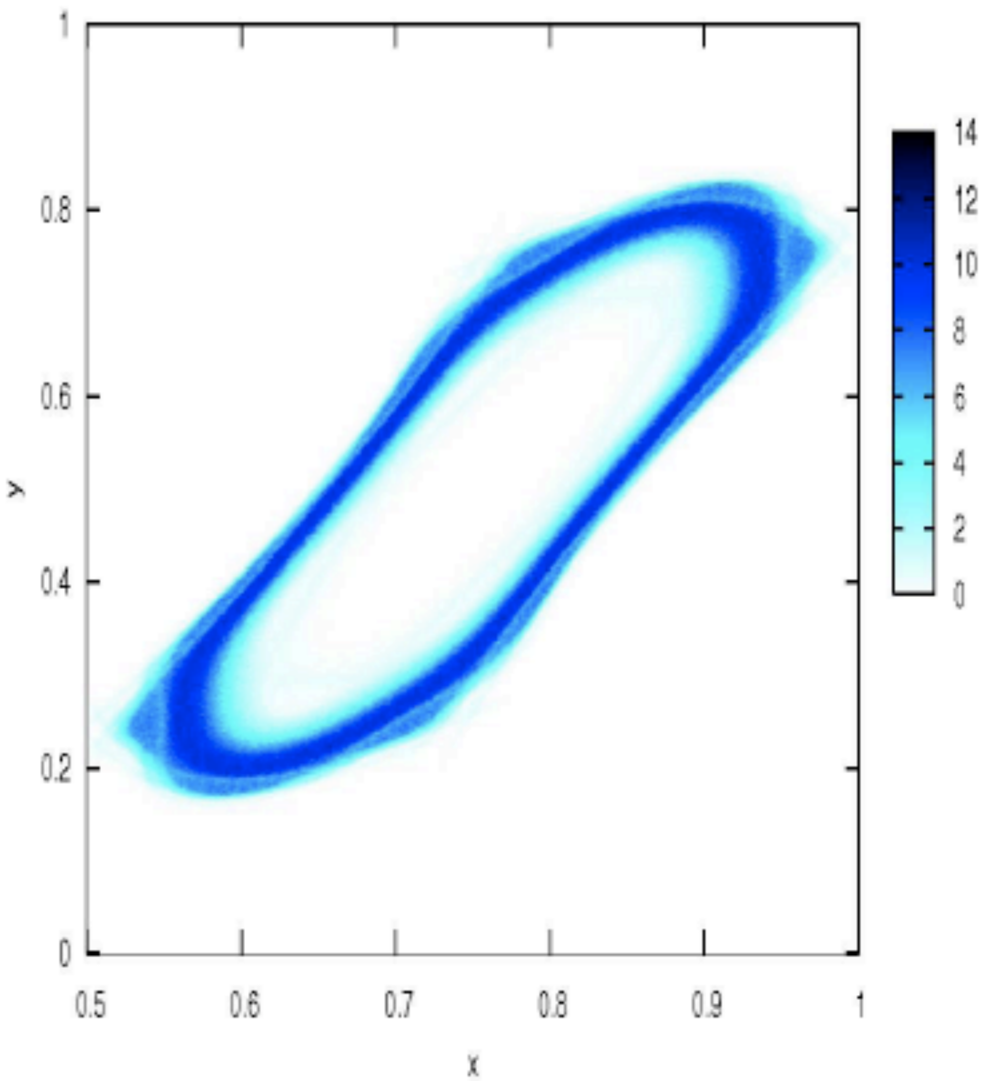

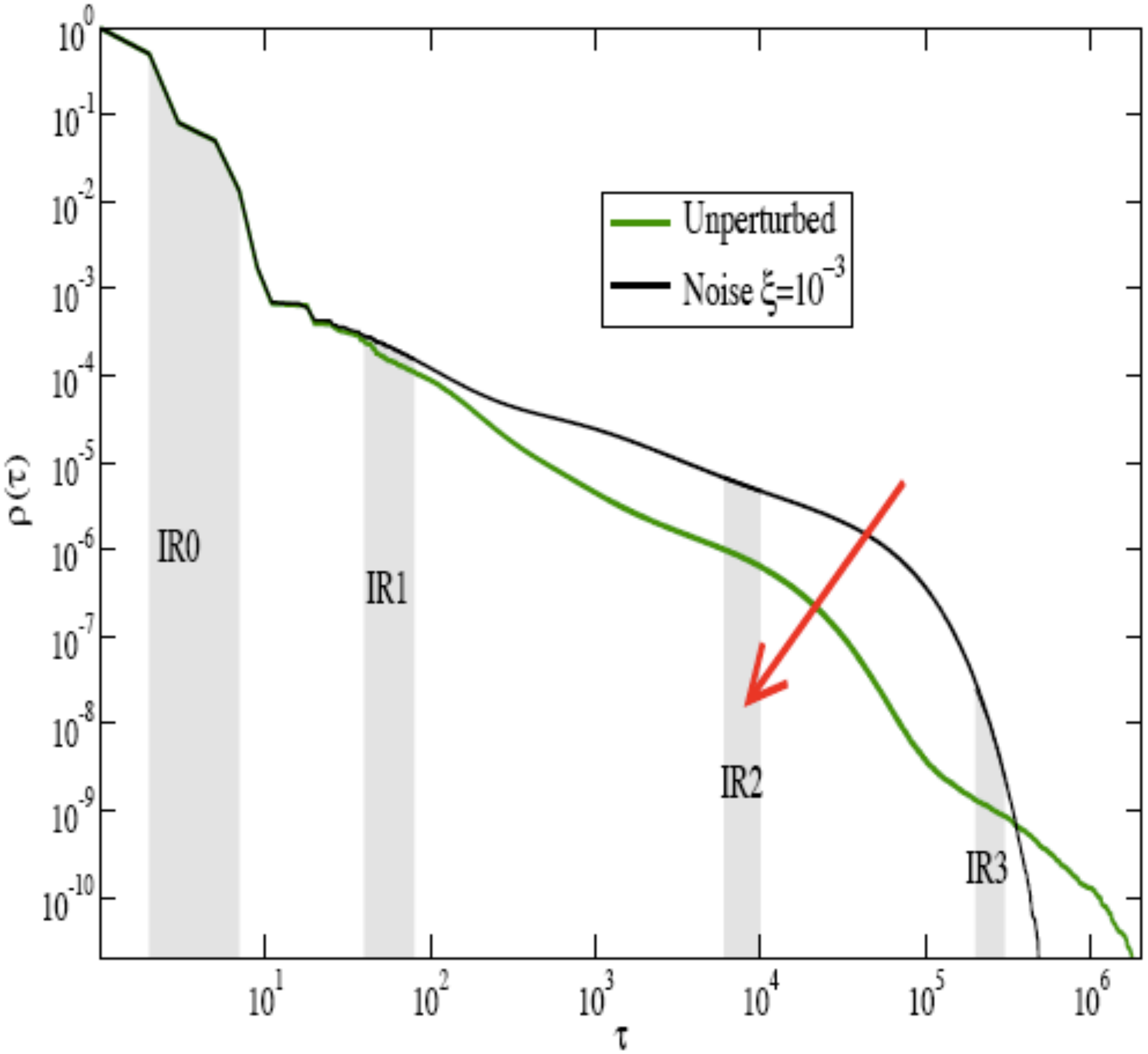

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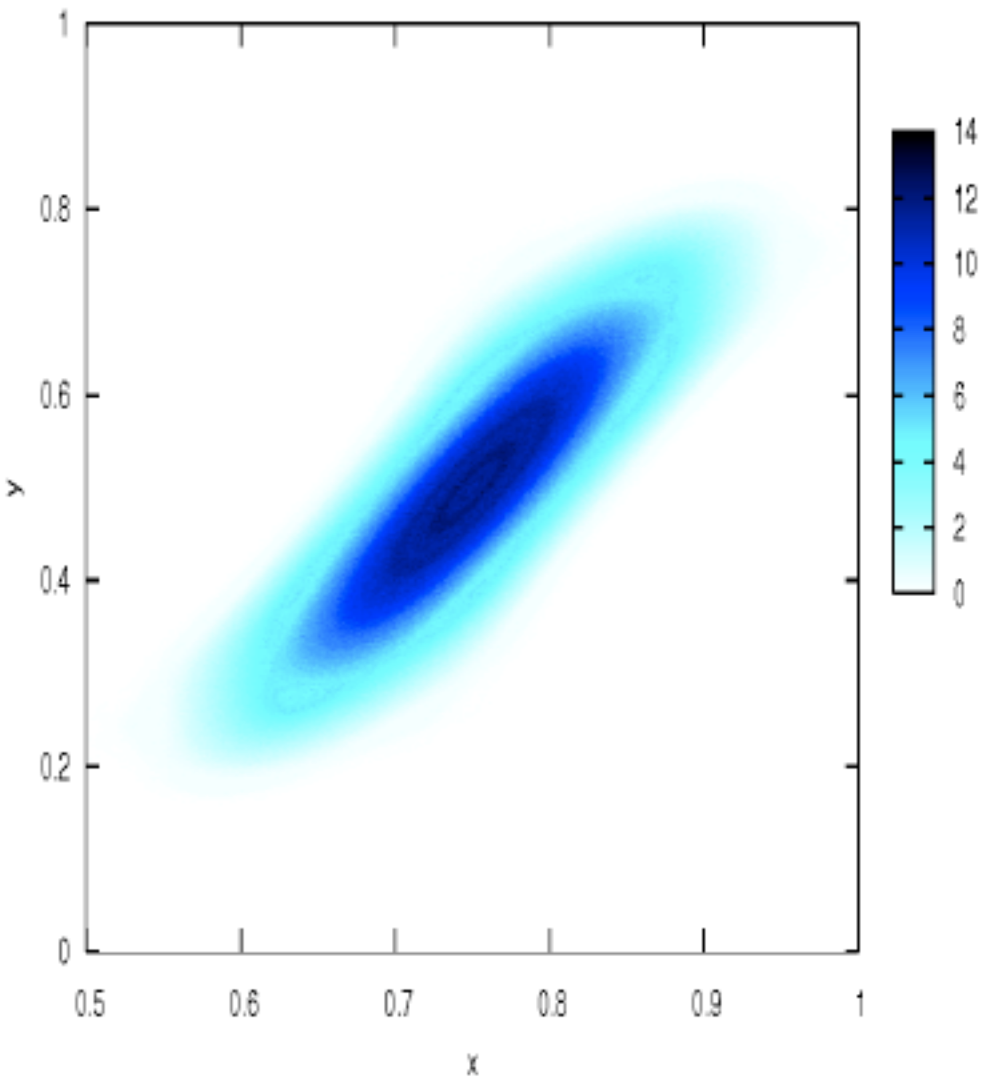
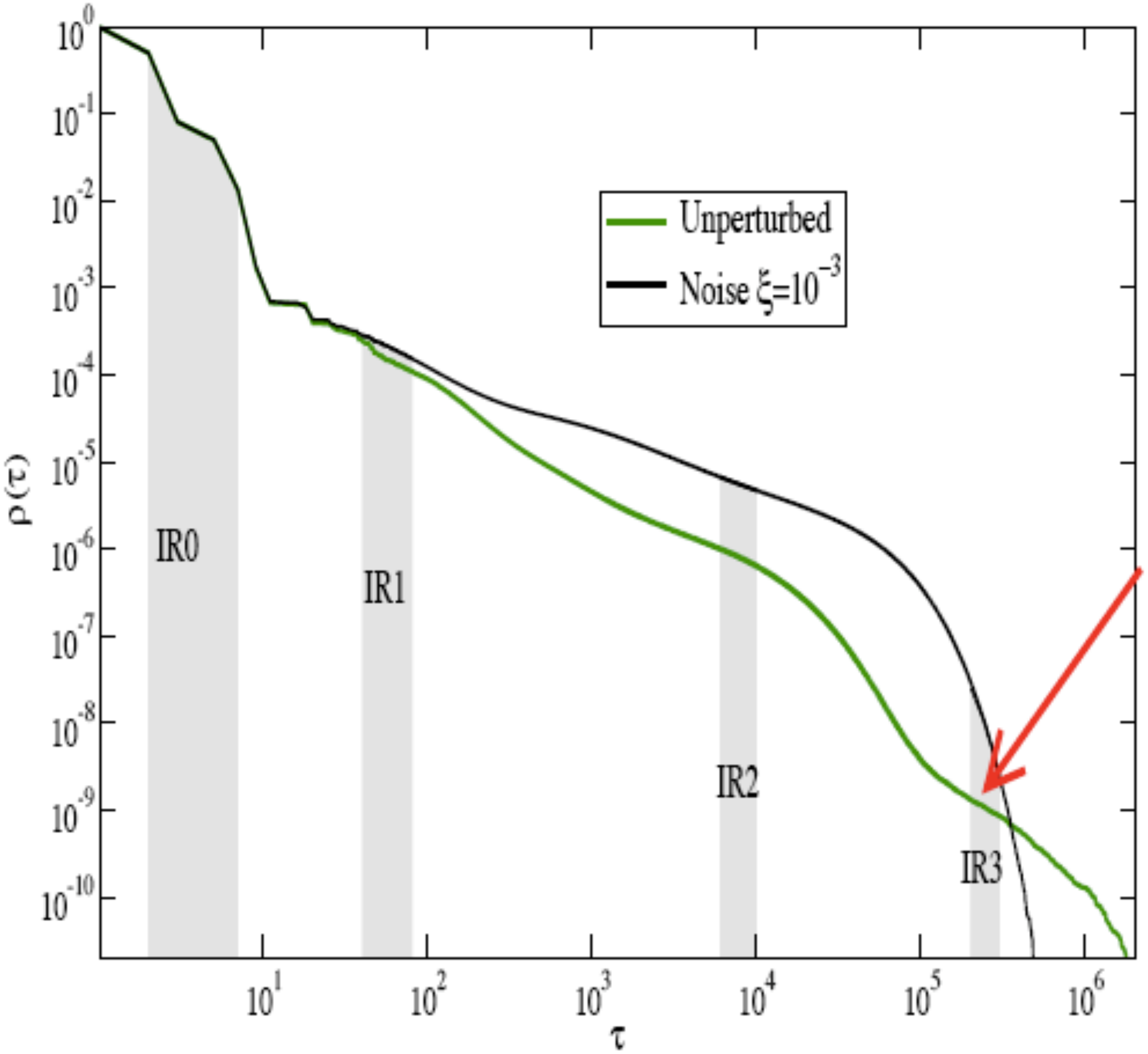
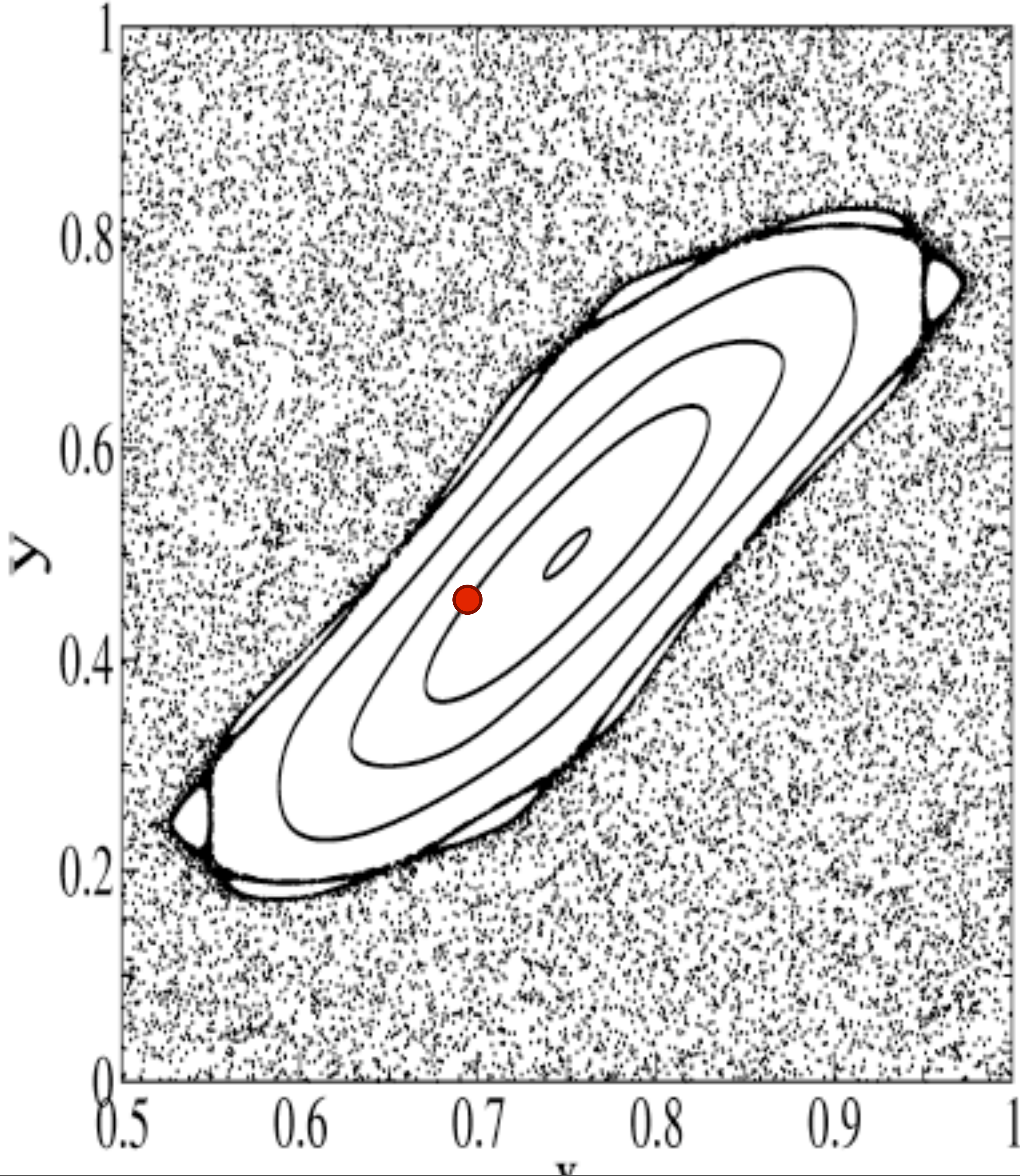
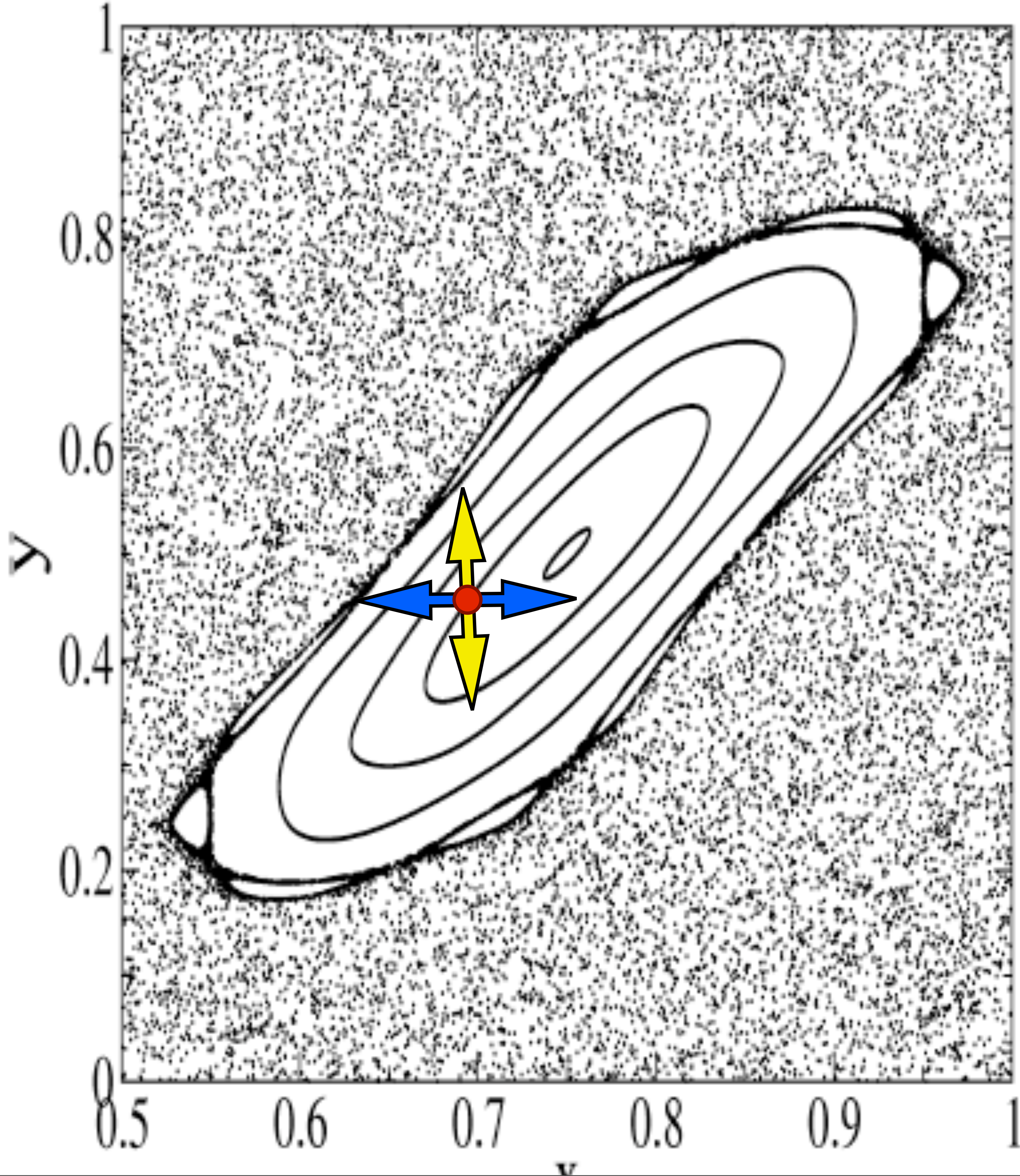




RW theory



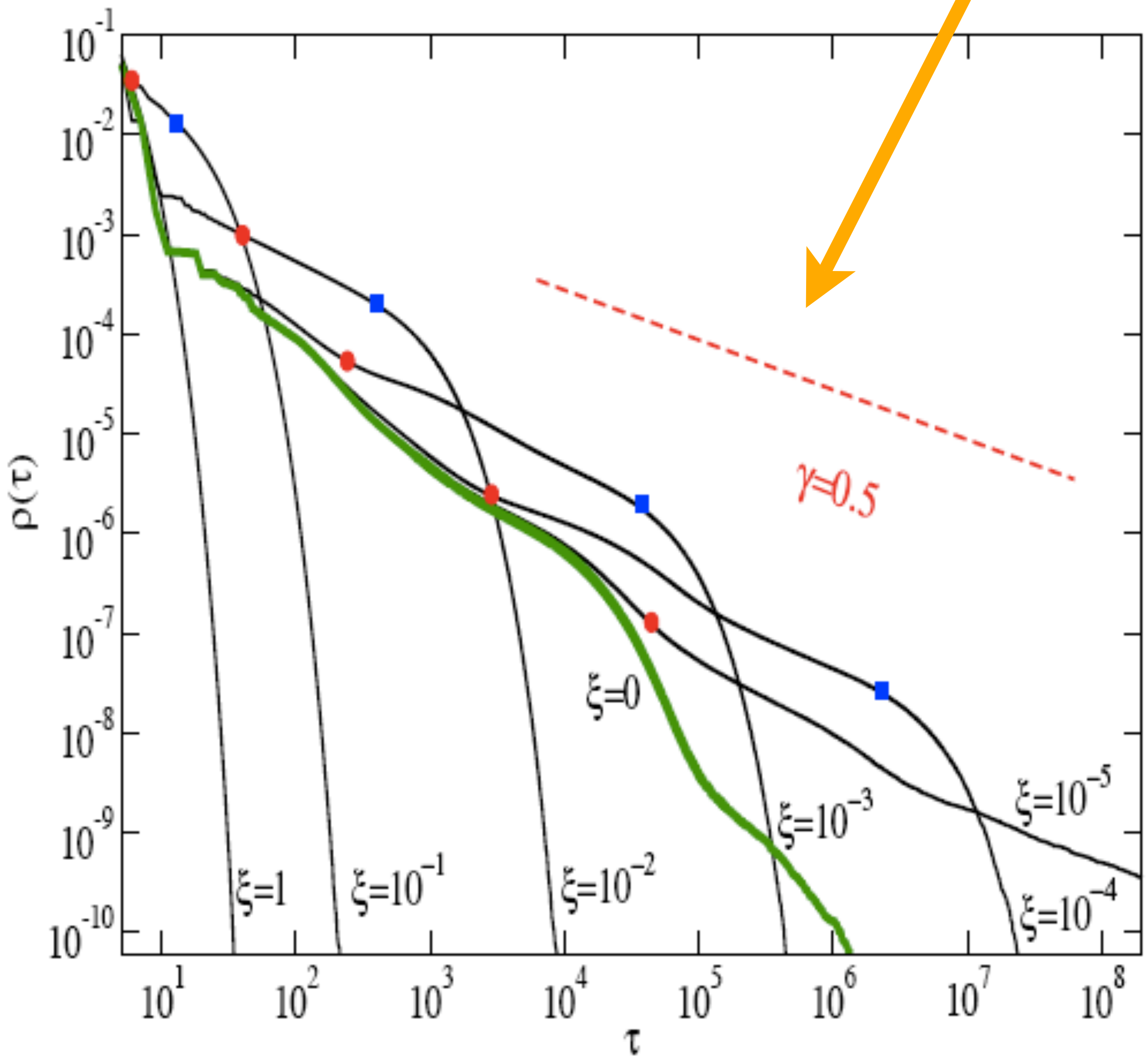
RW theory



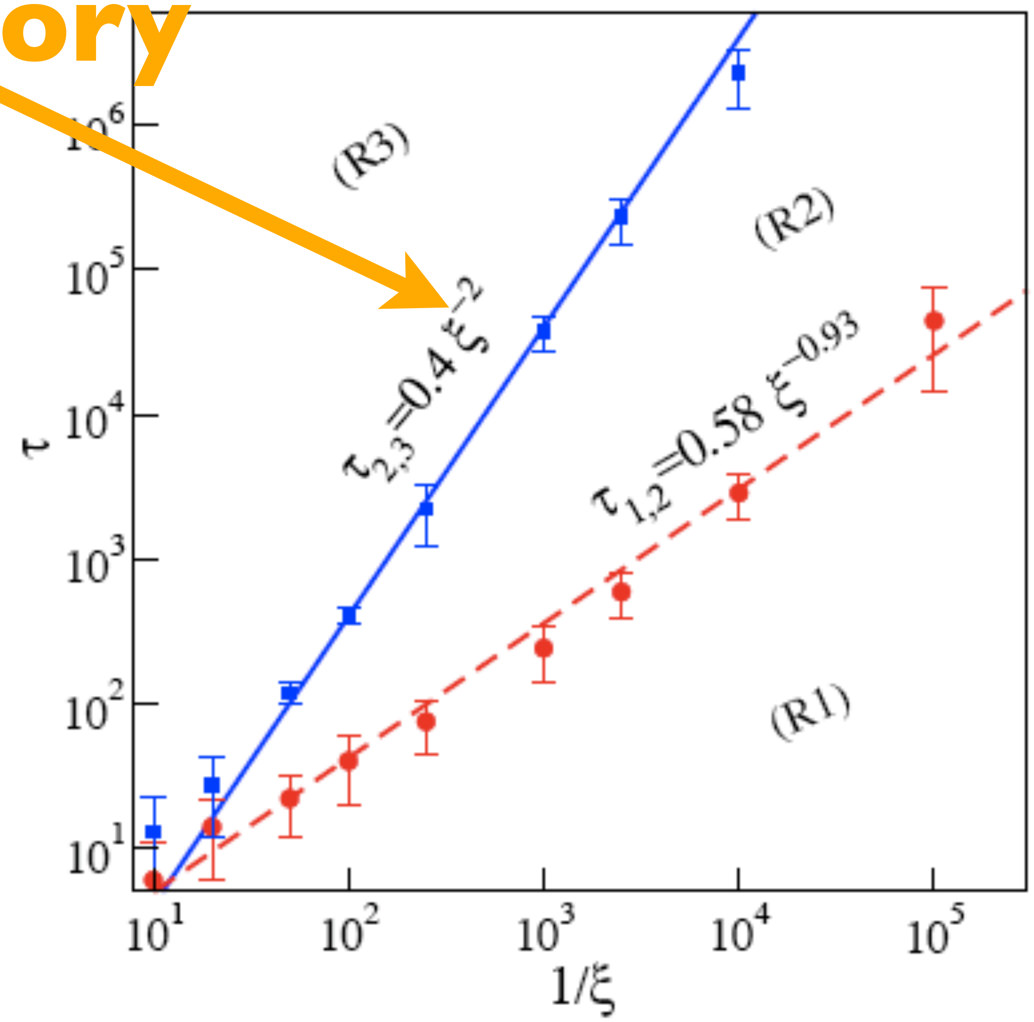
2.2 Noise perturbation

RW theory

Dependence on the noise intensity ξ



[E.G.A. and H. Kantz, Europhys. Lett. 07]

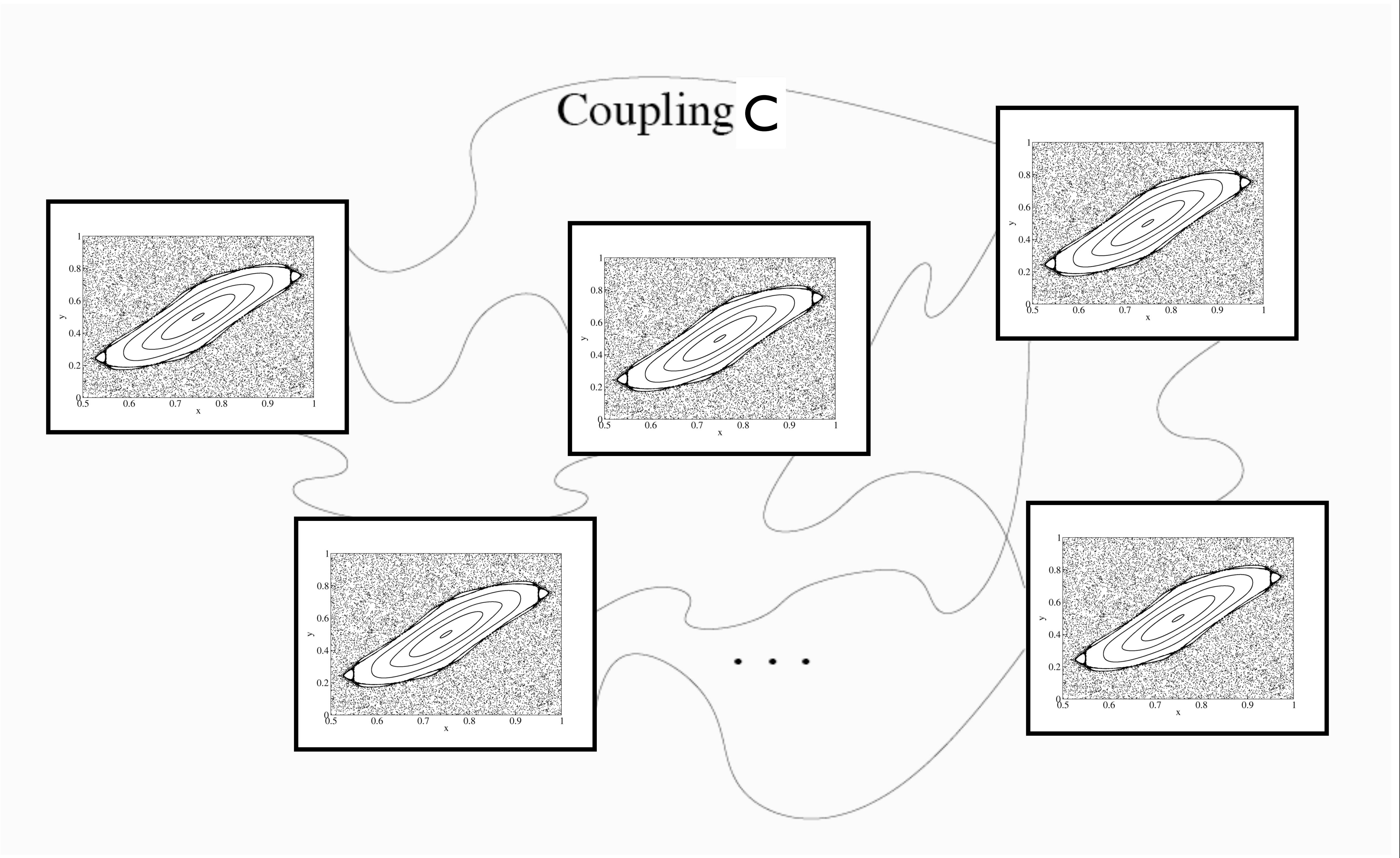


- Intermediate trapping:
  RW inside the island $\Rightarrow \gamma = 0.5$.
- $\tau_{1,2} \sim \xi^{-\beta}, \quad \beta \lessapprox 1$
  [Boffetta et al. 03].
- $\tau_{2,3} \sim \xi^{-2}$
  RW in a finite domain.

Coupled standard maps:

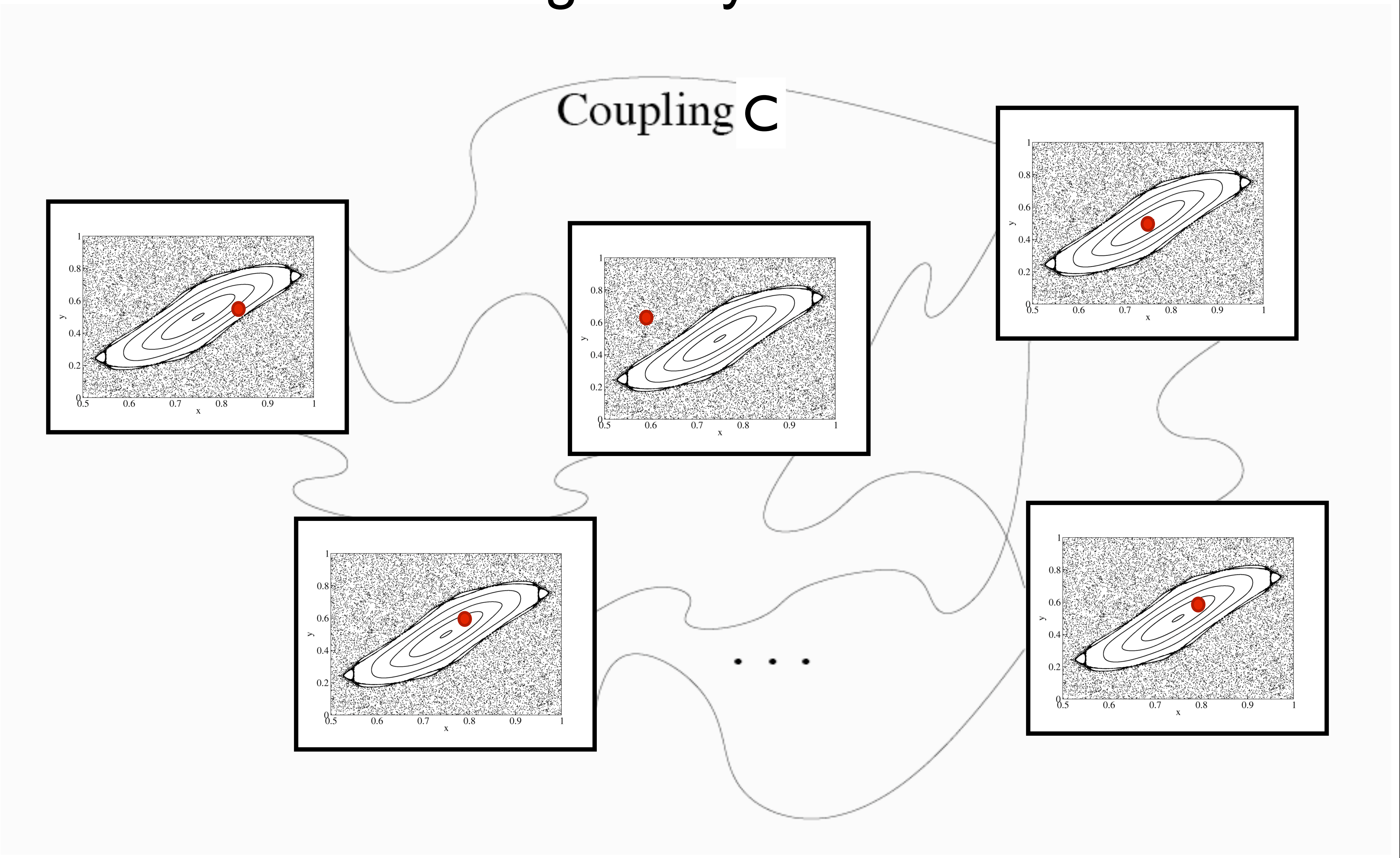
2.1 Motivation / model
2.2 Noise perturbation
→ 2.3 High dimensional

Coupled symplectic maps model



Coupled symplectic maps model

Ergodicity?



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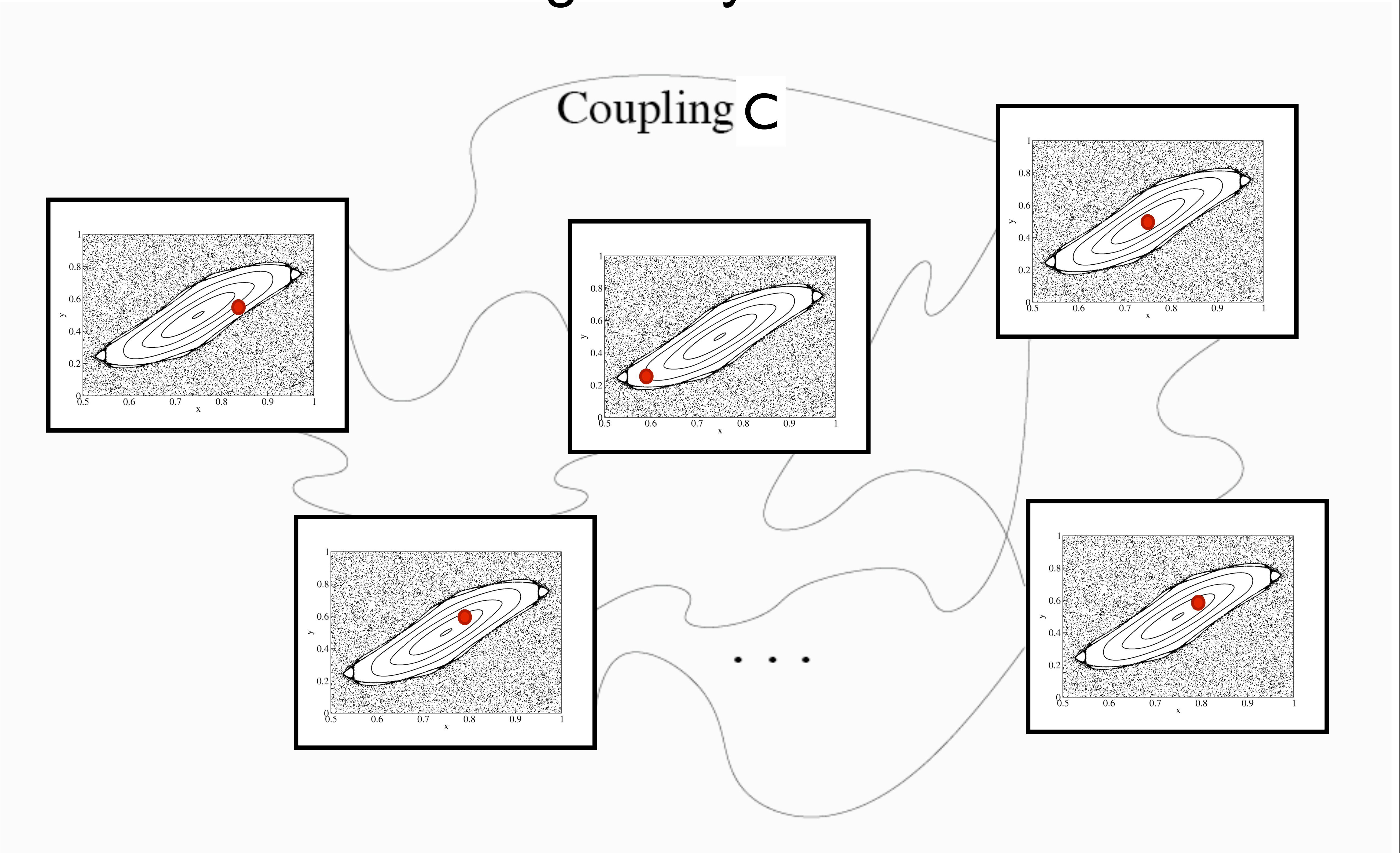
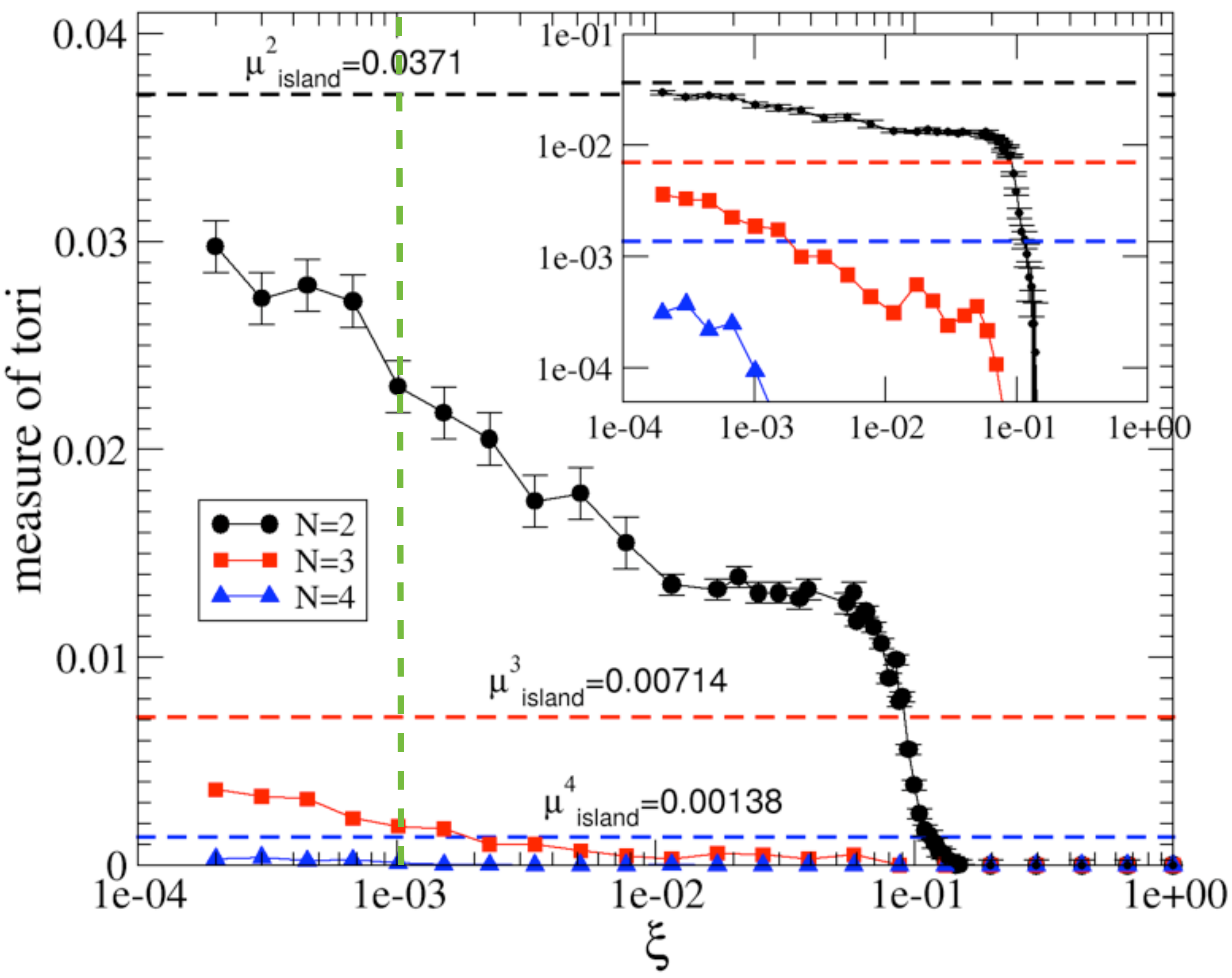
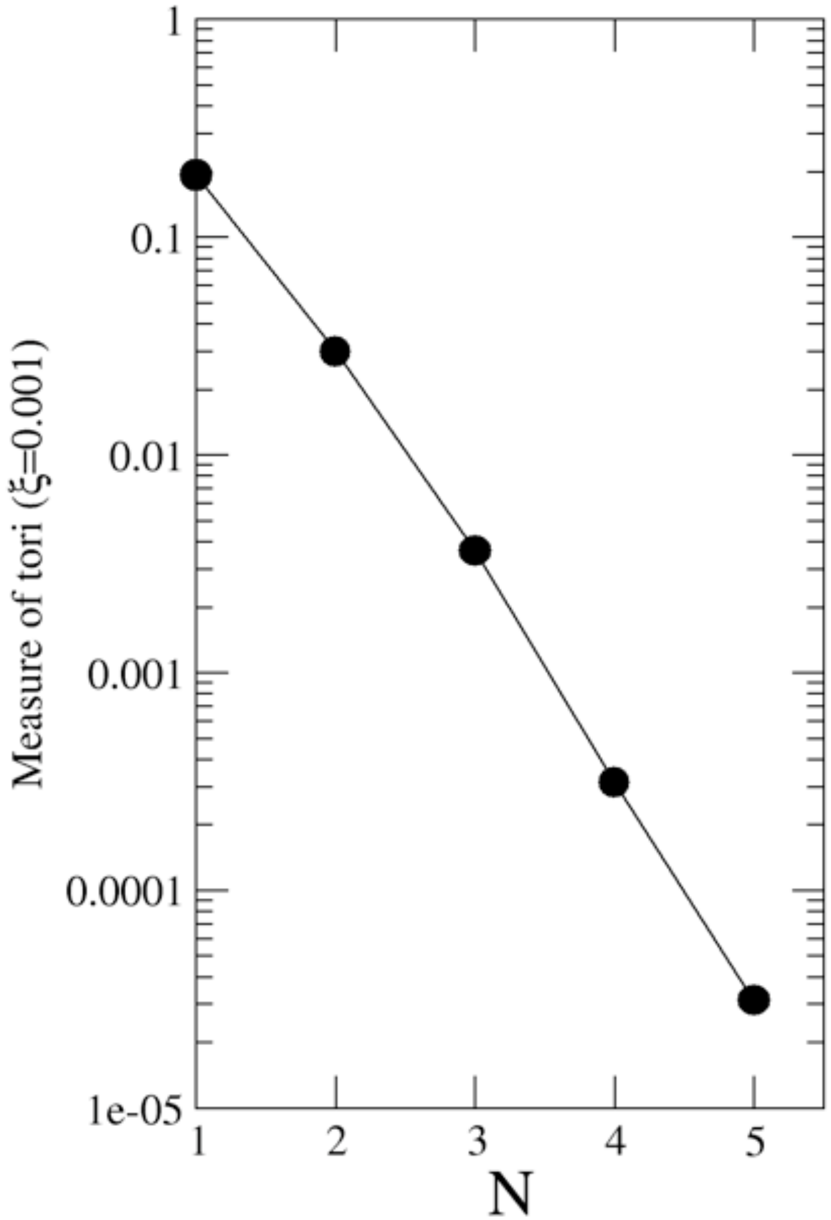


Coupled symplectic maps model

Ergodicity?





Coupled symplectic maps model

1. Ergodicity, i.e., negligible measure of regular components ✔

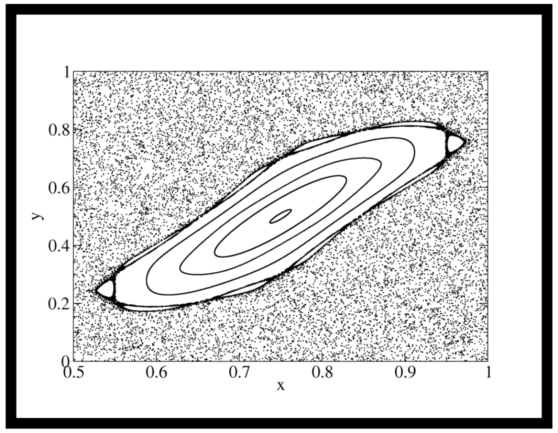
↓ ✗ e.g., zero measure sets on Bunimovich stadium Billiards

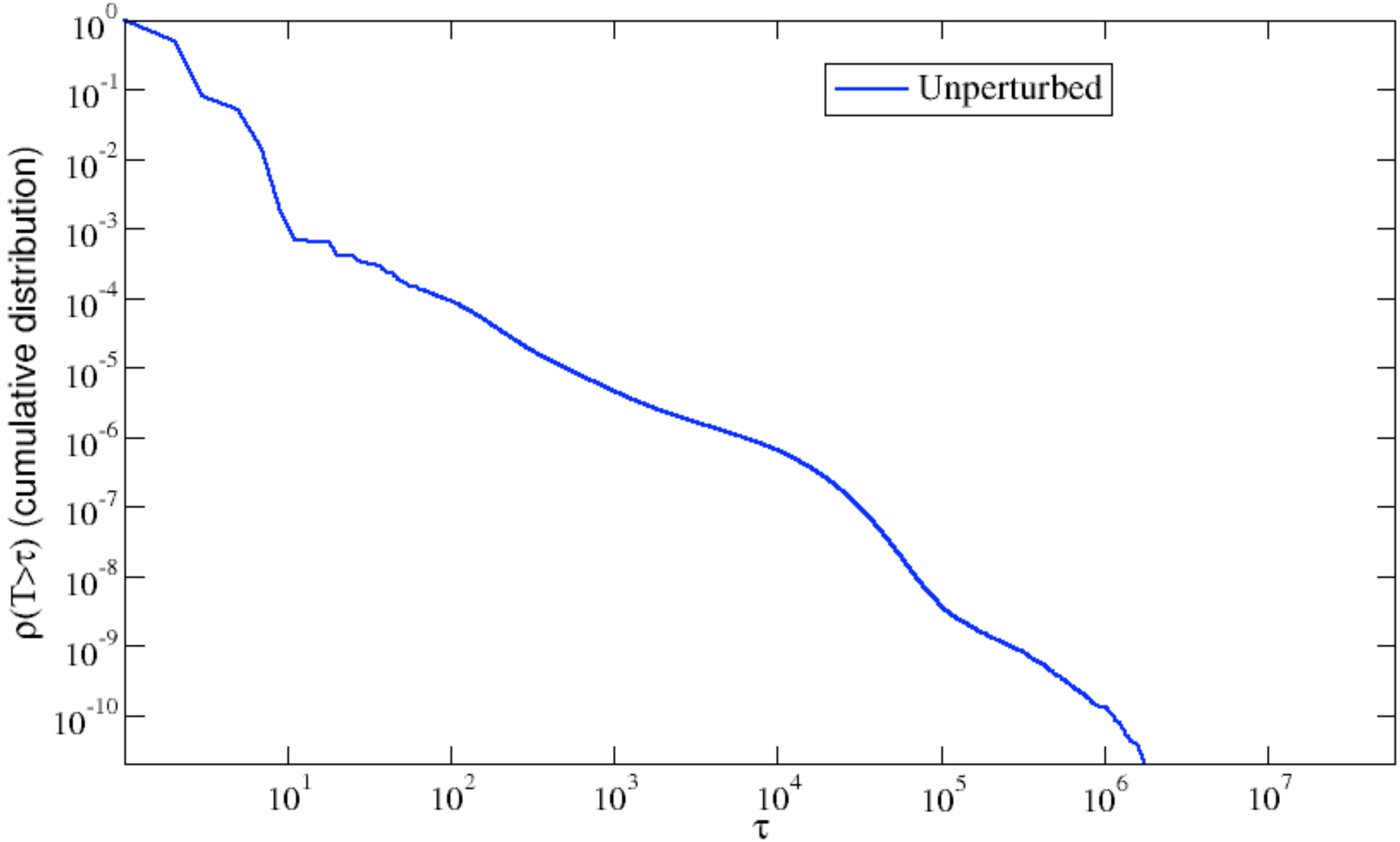
2. Strong mixing, i.e., fast decay of correlations ?

N=2-5 show power-law behavior [Kantz, Grassberger (1987), Ding, Bountis, Ott (1990)]

Coupled symplectic maps model

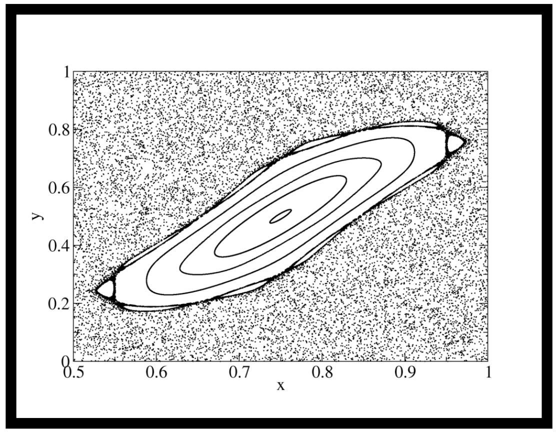
Strong mixing?

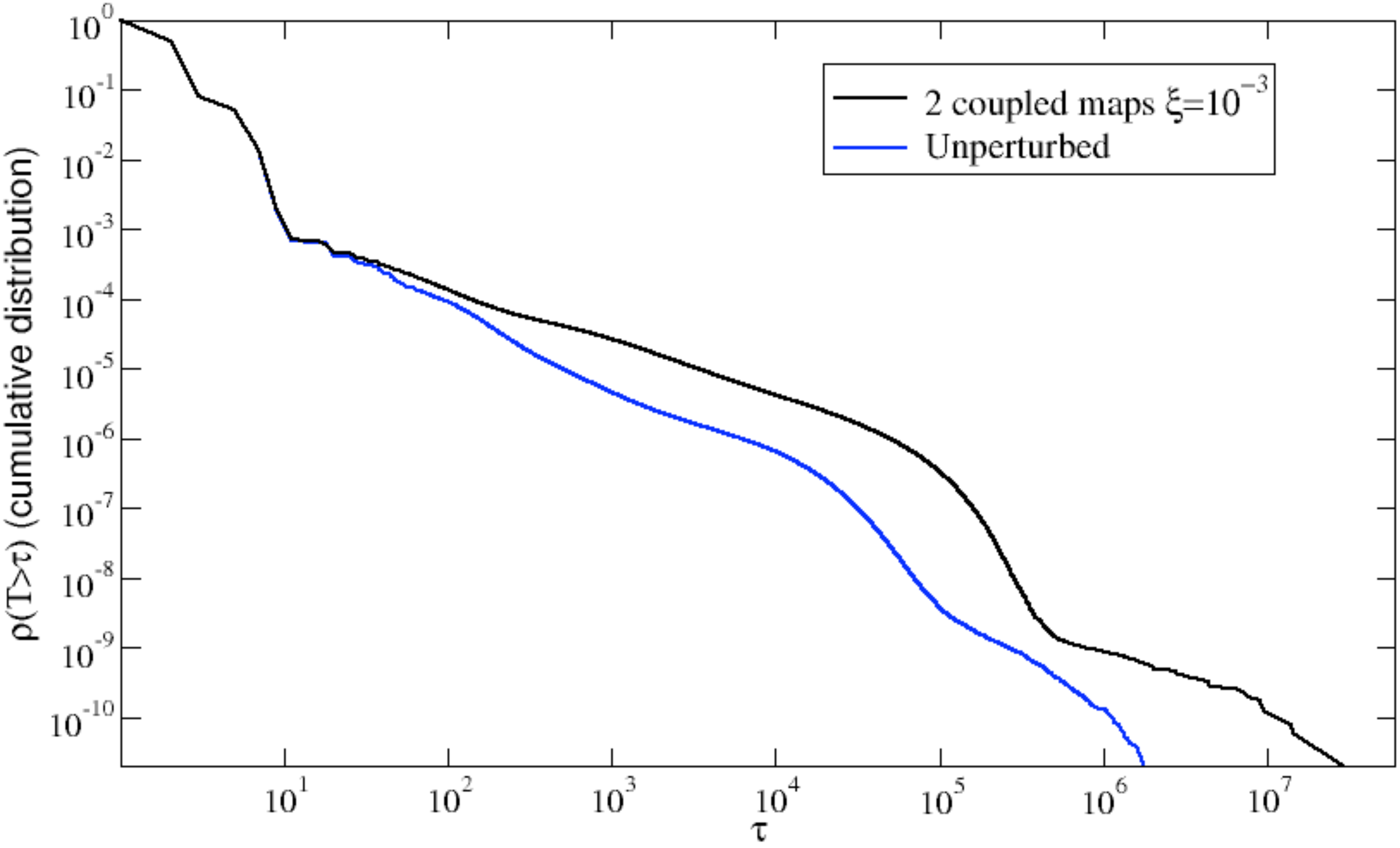




Coupled symplectic maps model

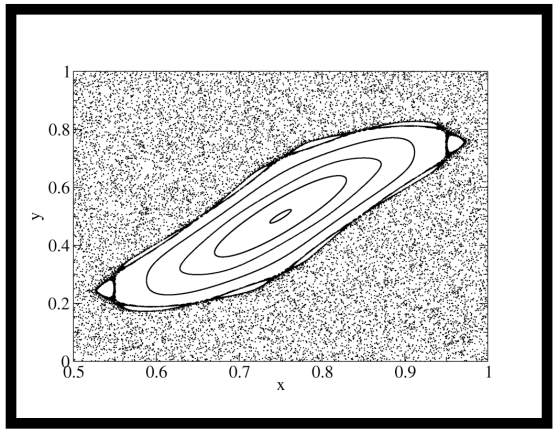
Strong mixing?

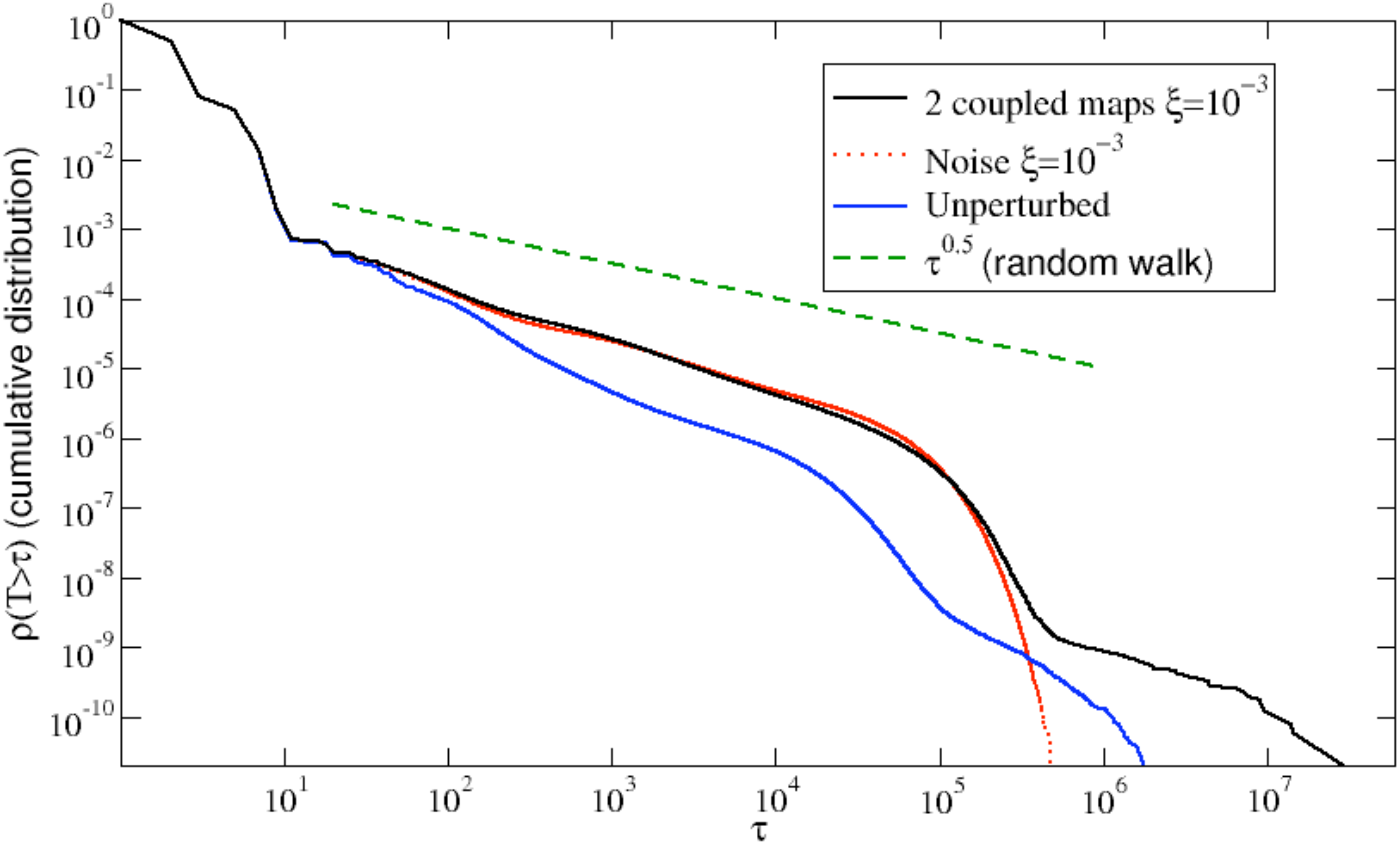




Coupled symplectic maps model

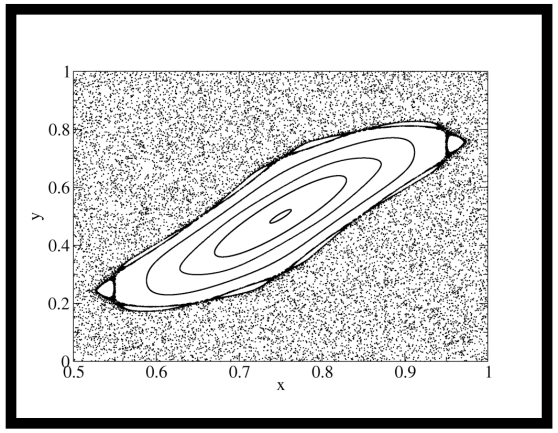
Strong mixing?

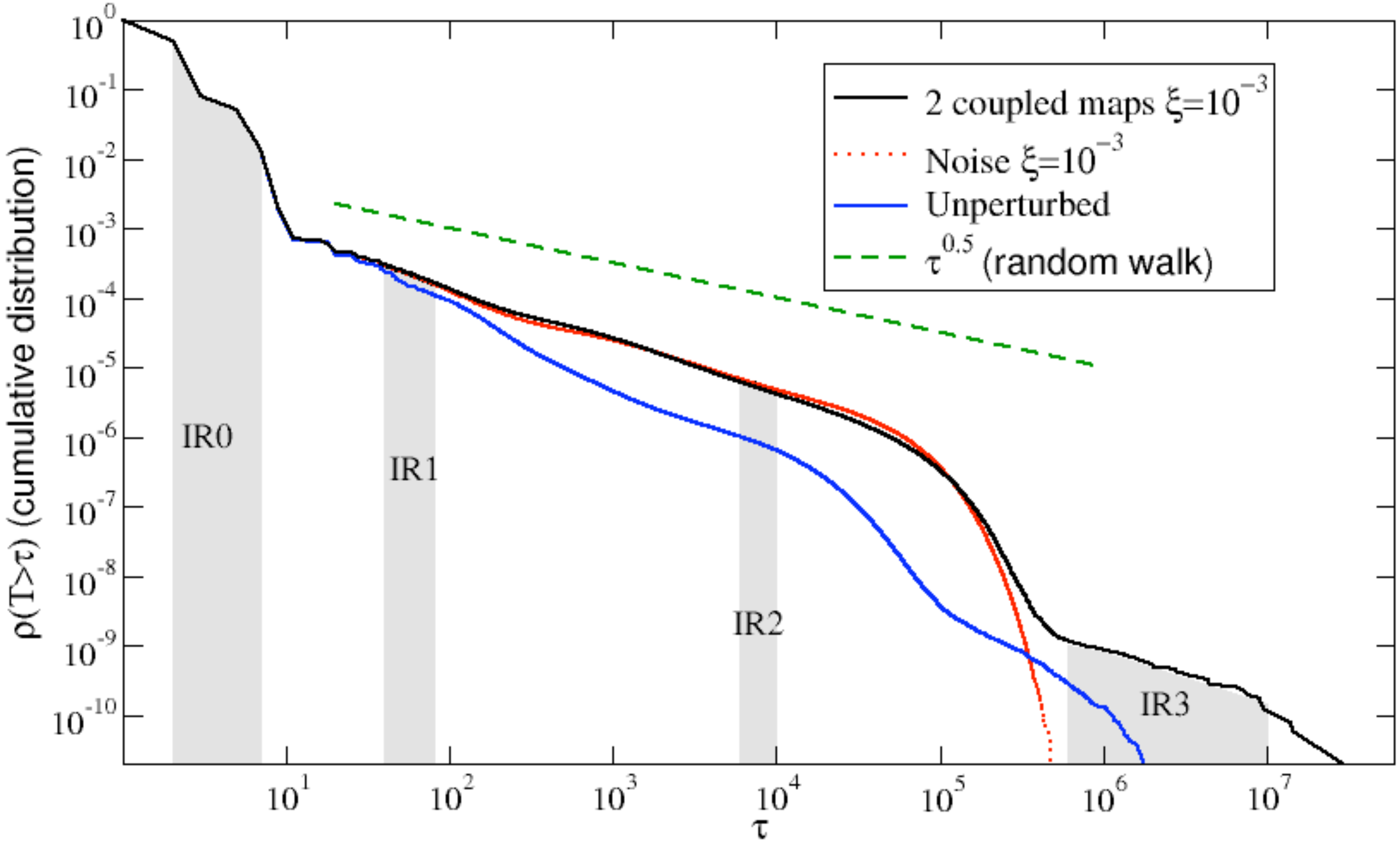




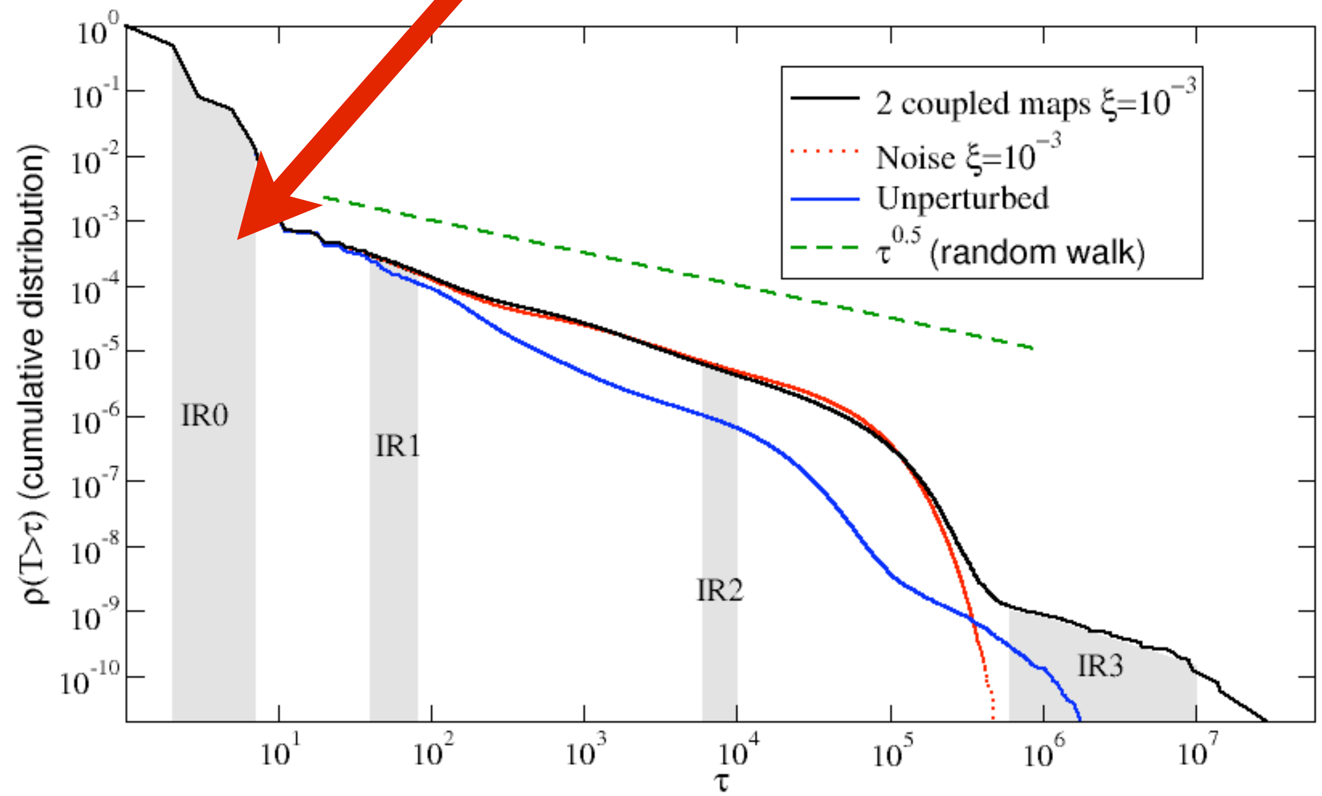
Coupled symplectic maps model

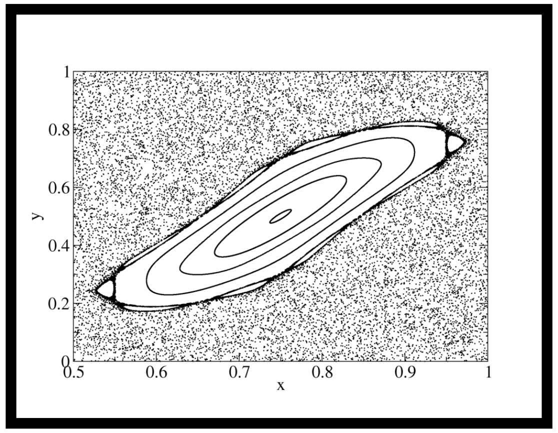
Strong mixing?

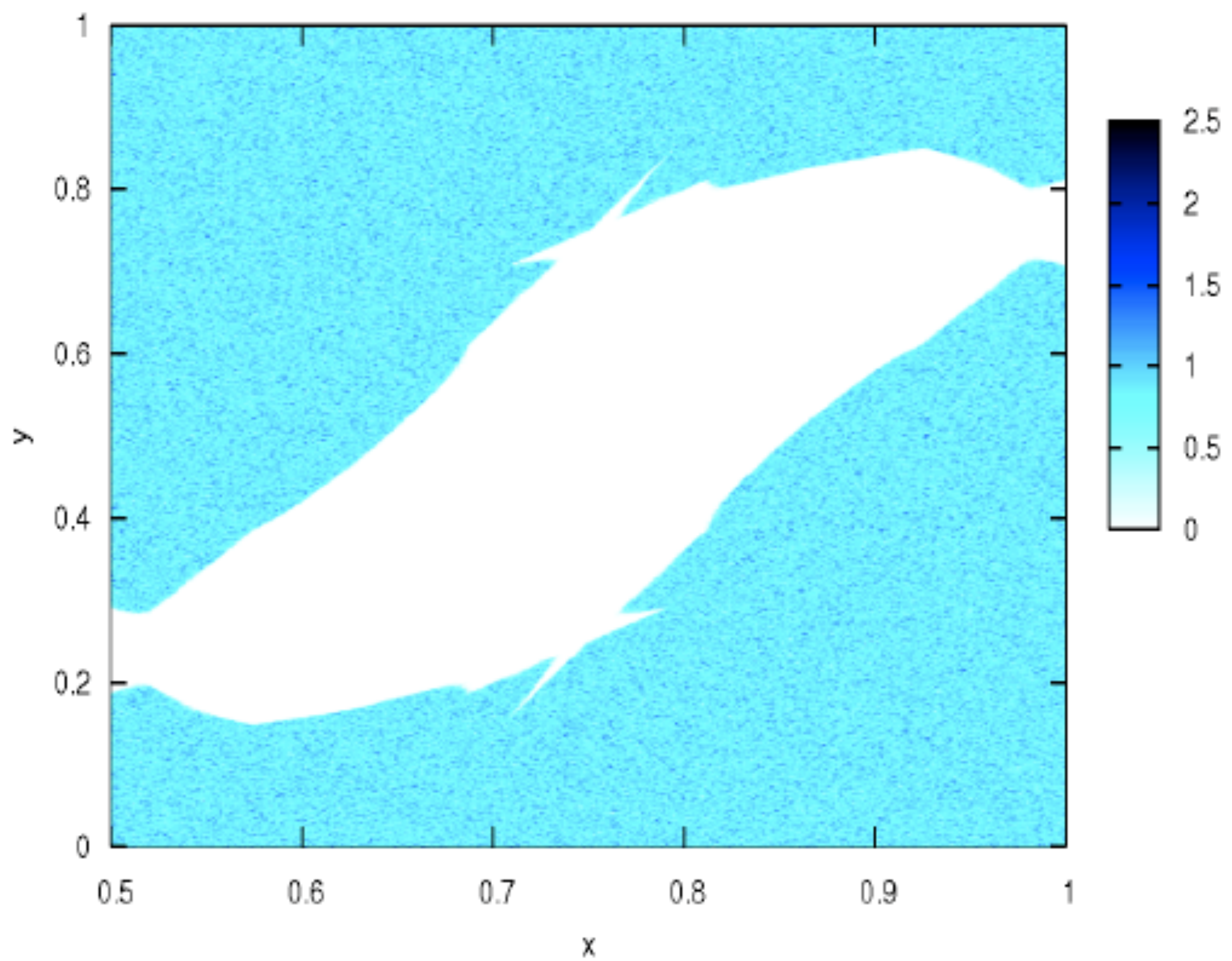


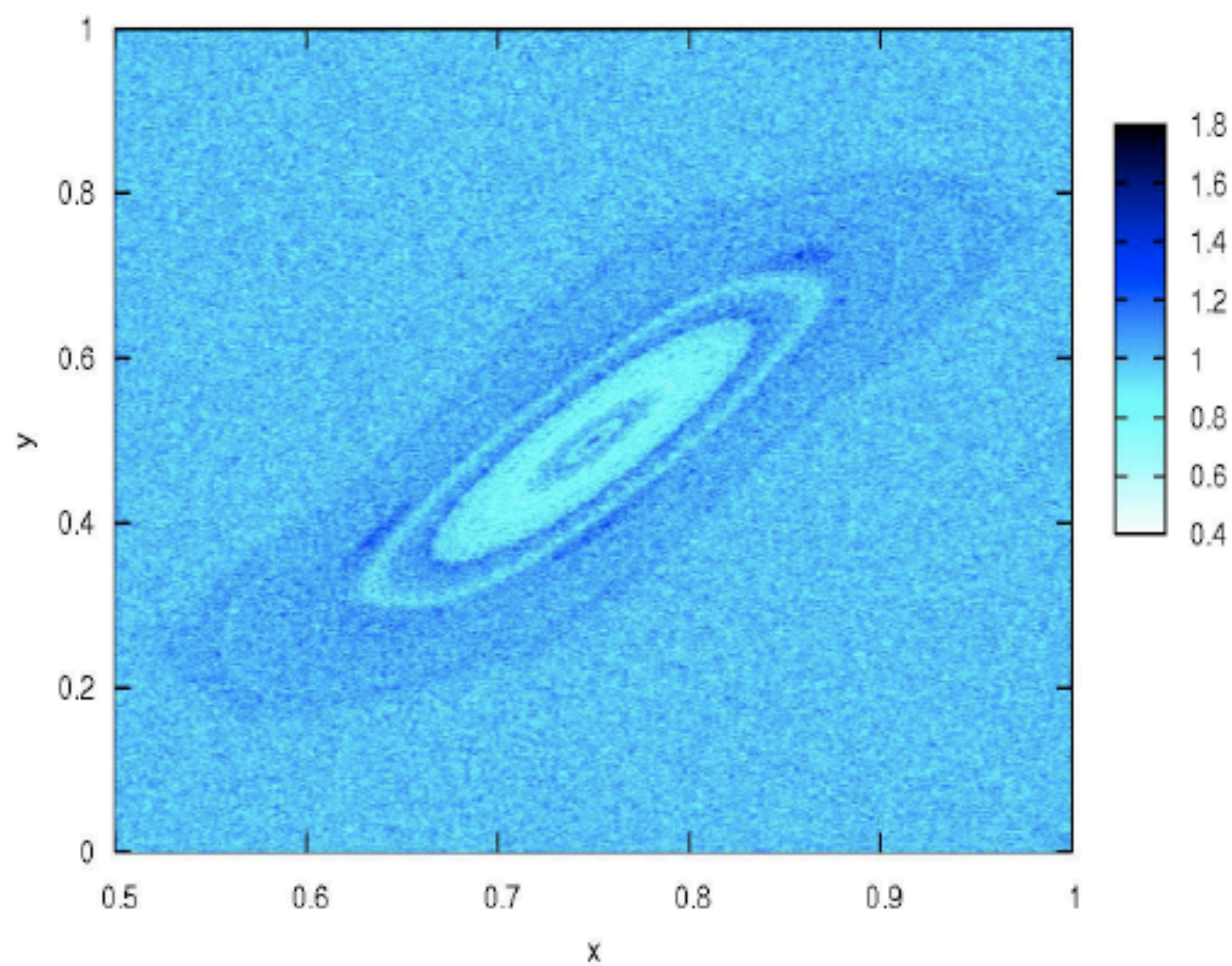


Coupled symplectic maps model

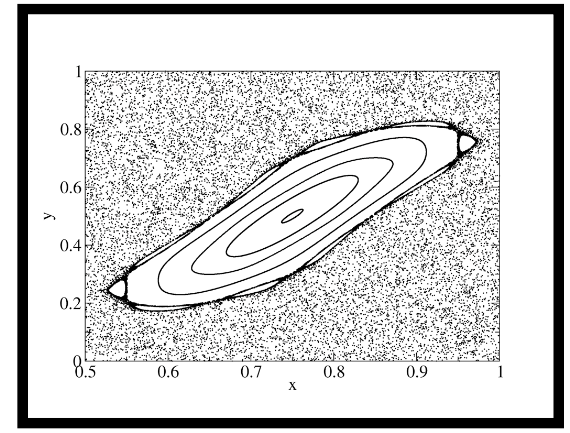


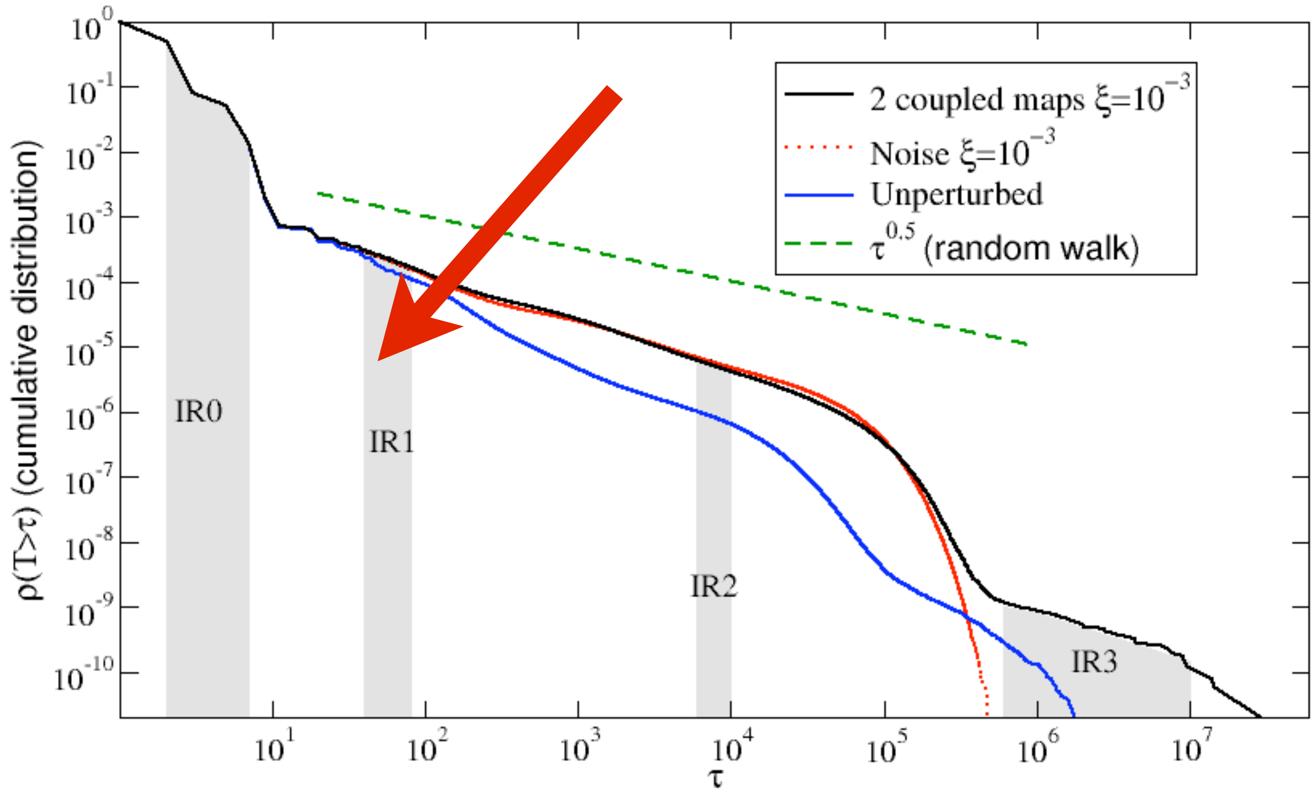


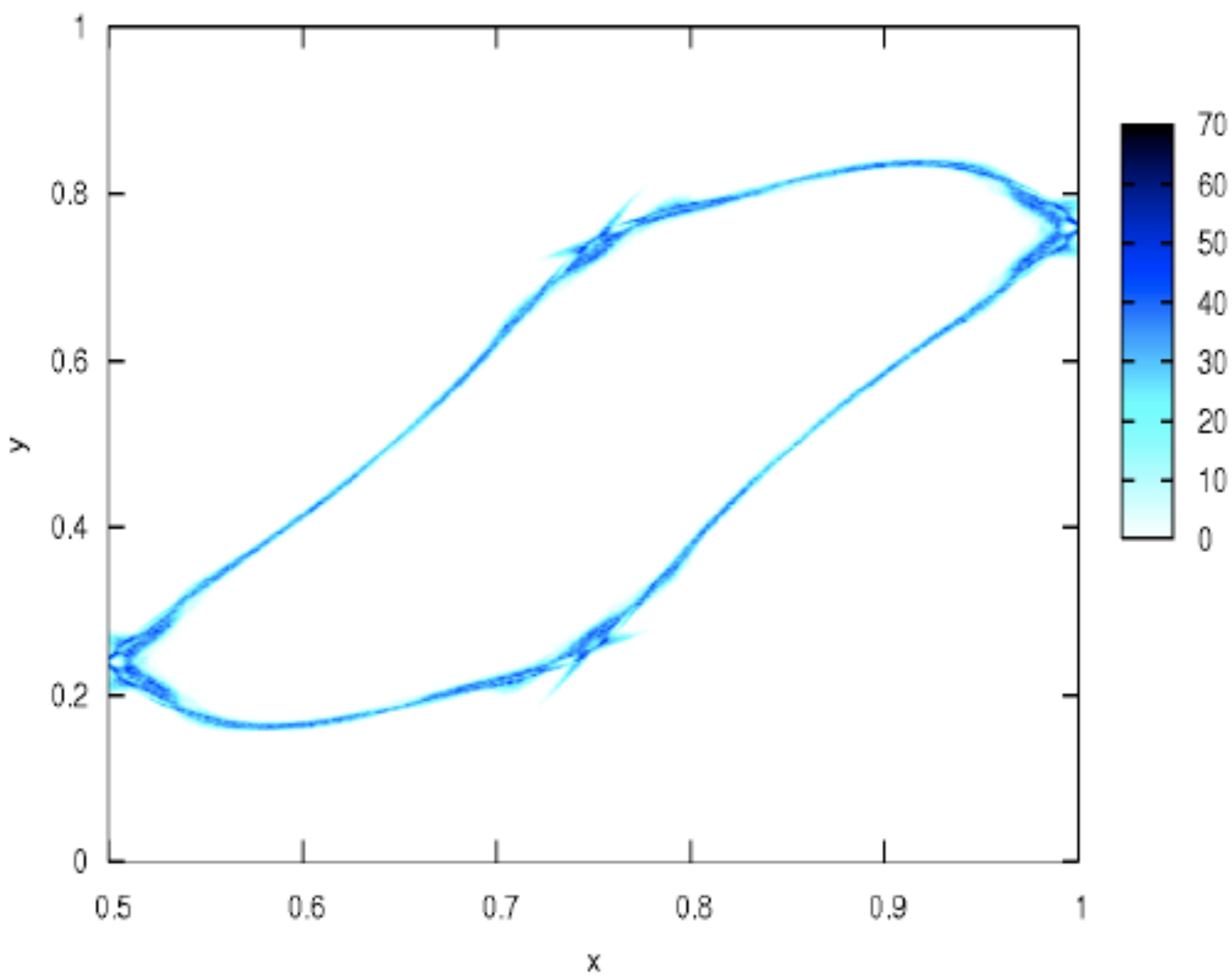


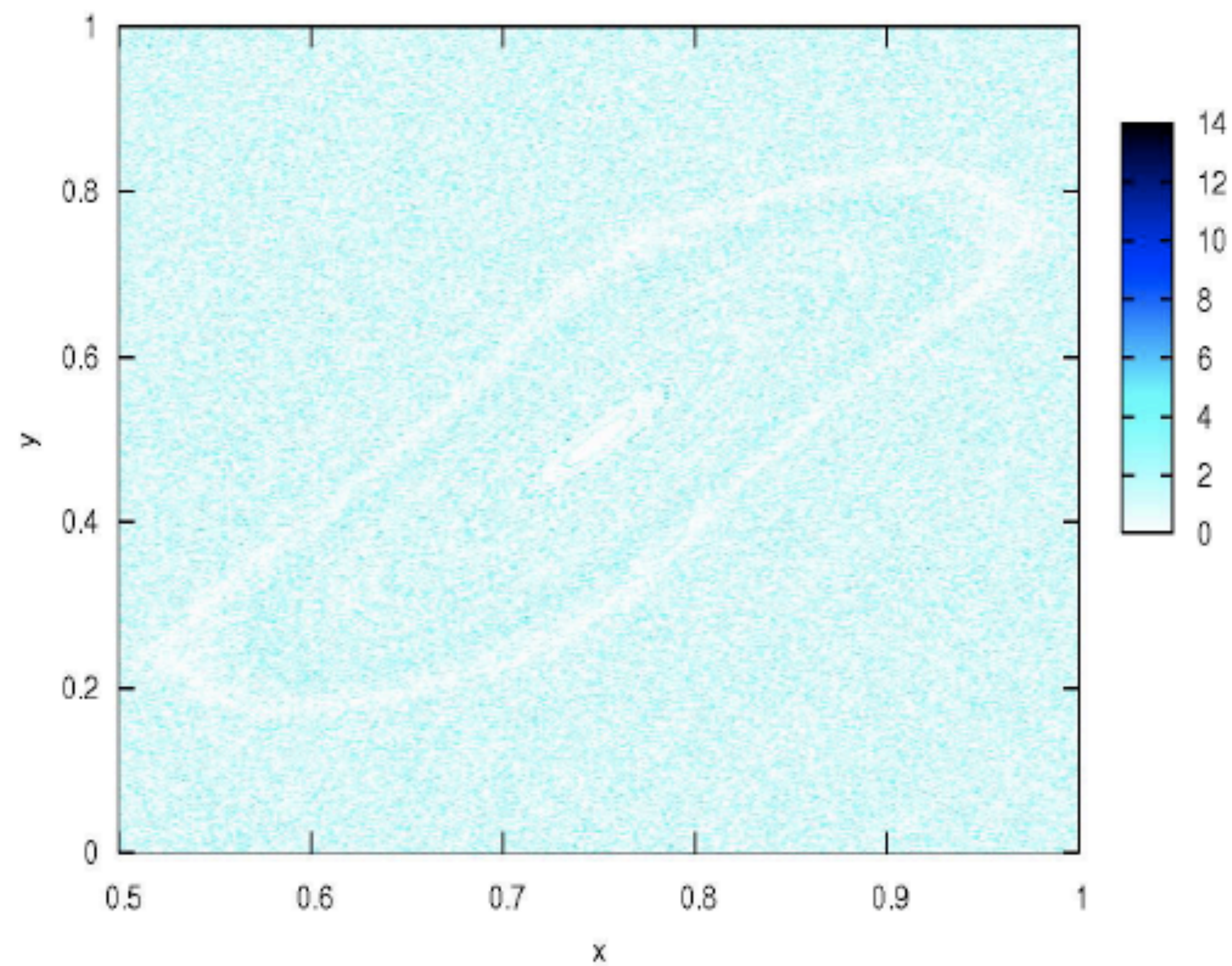


Coupled symplectic maps model

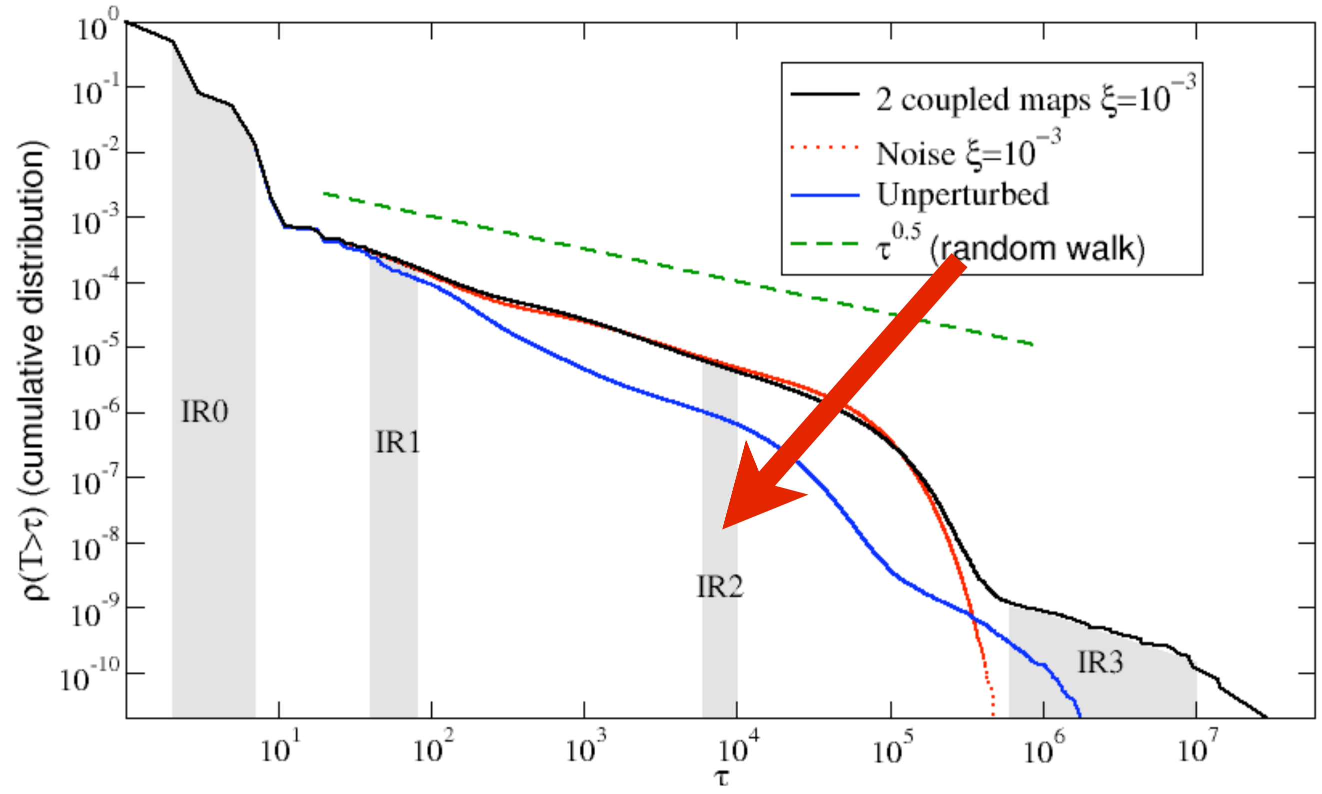


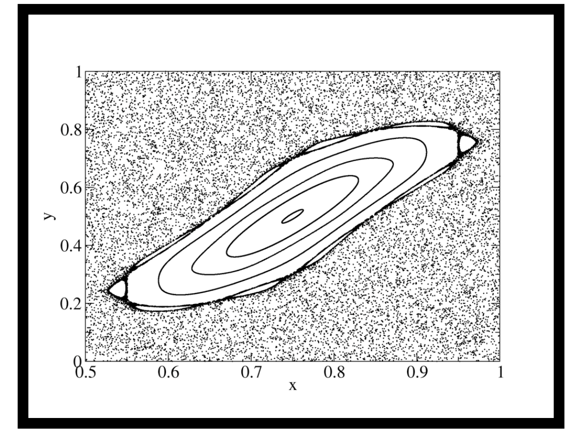


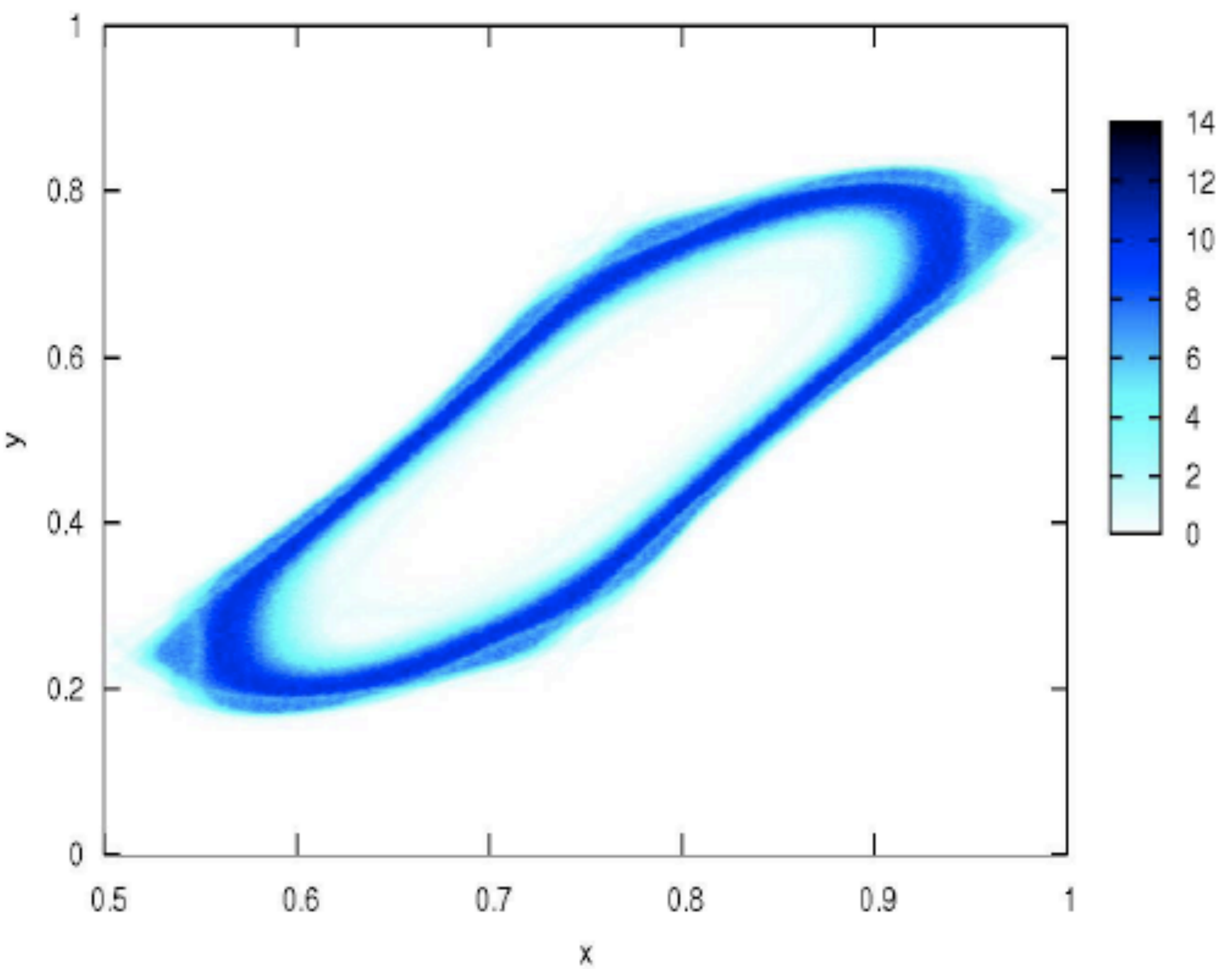


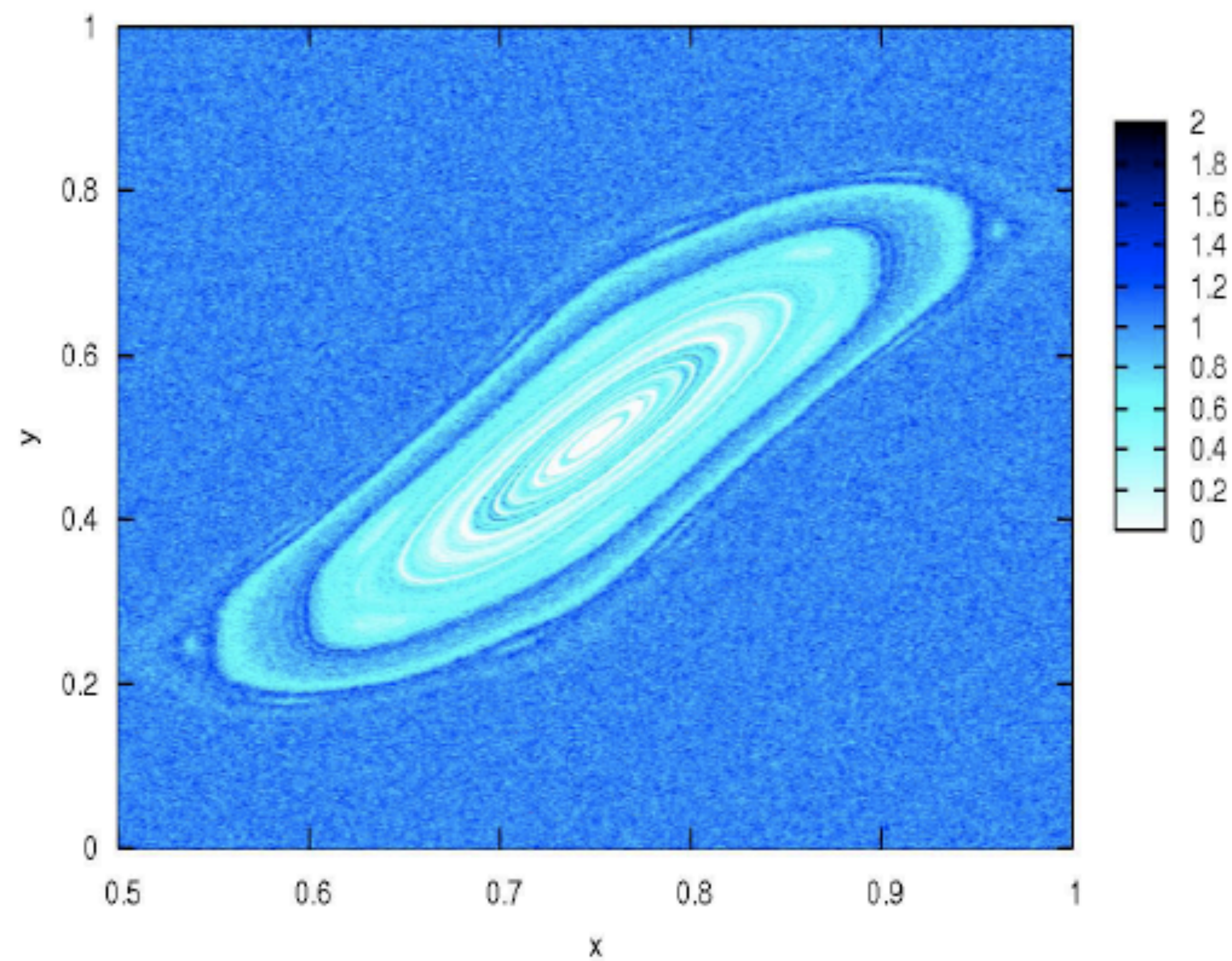


Coupled symplectic maps model

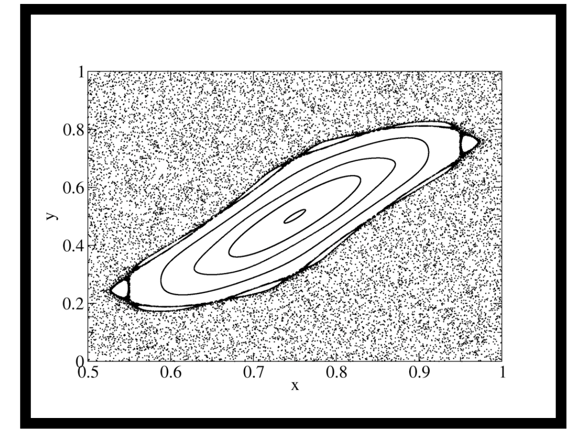


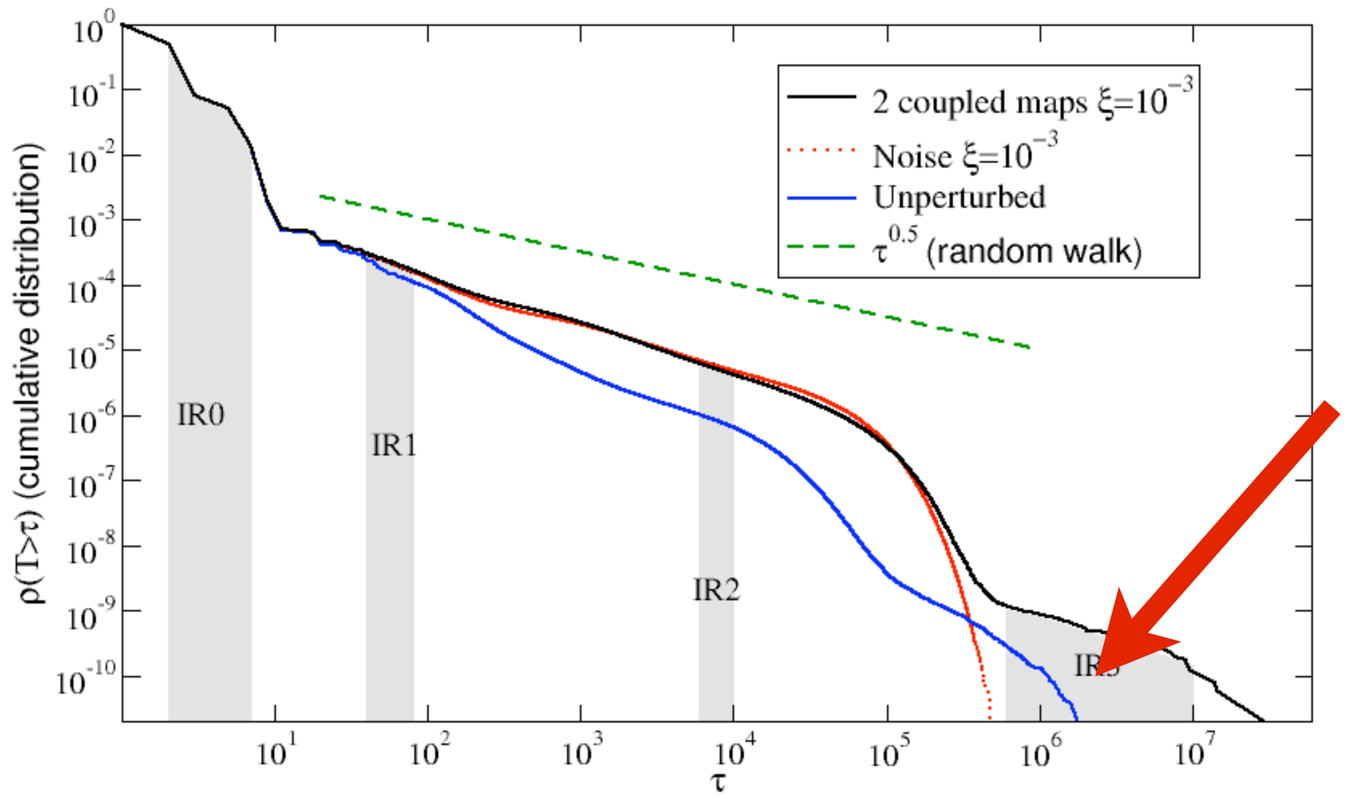


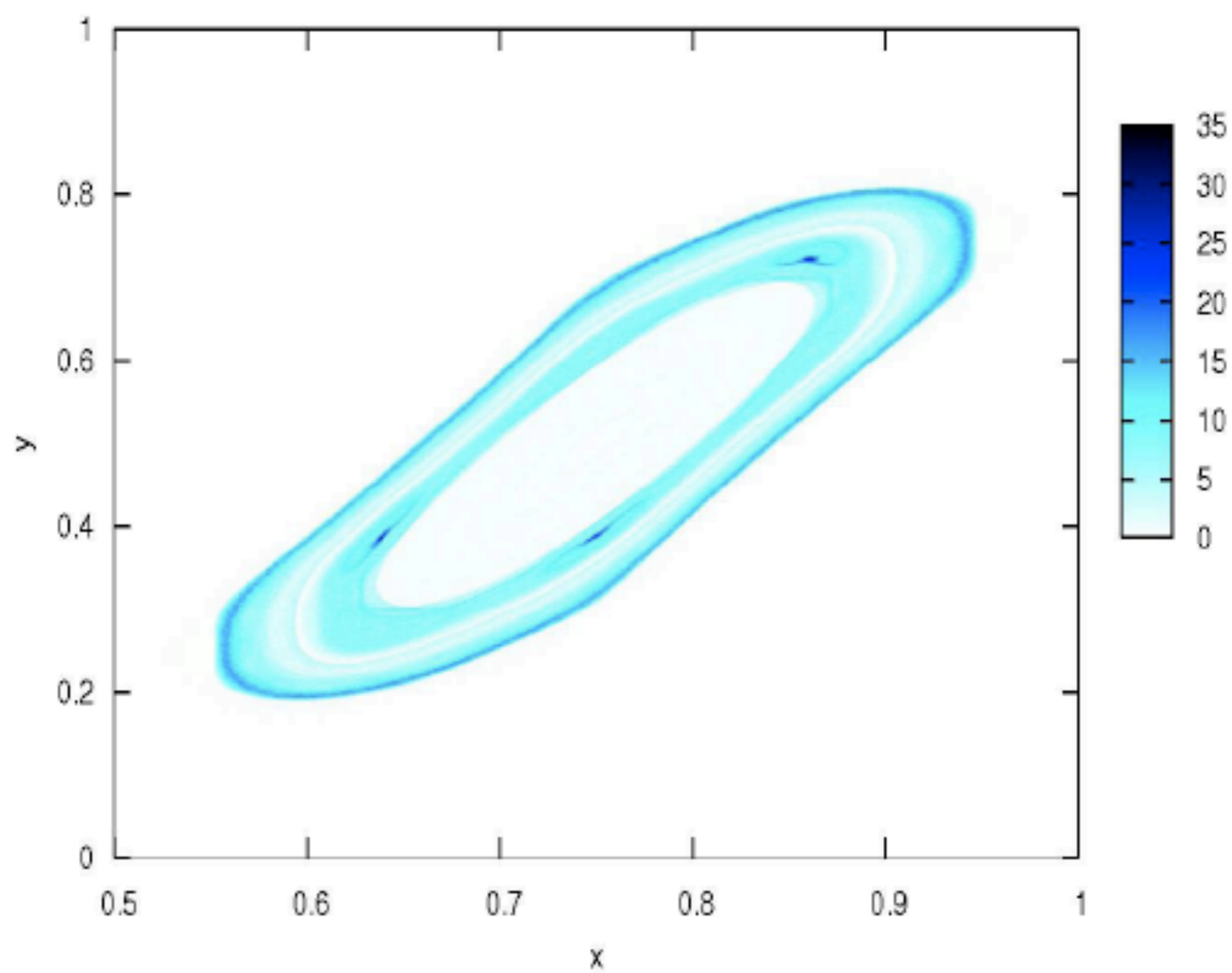


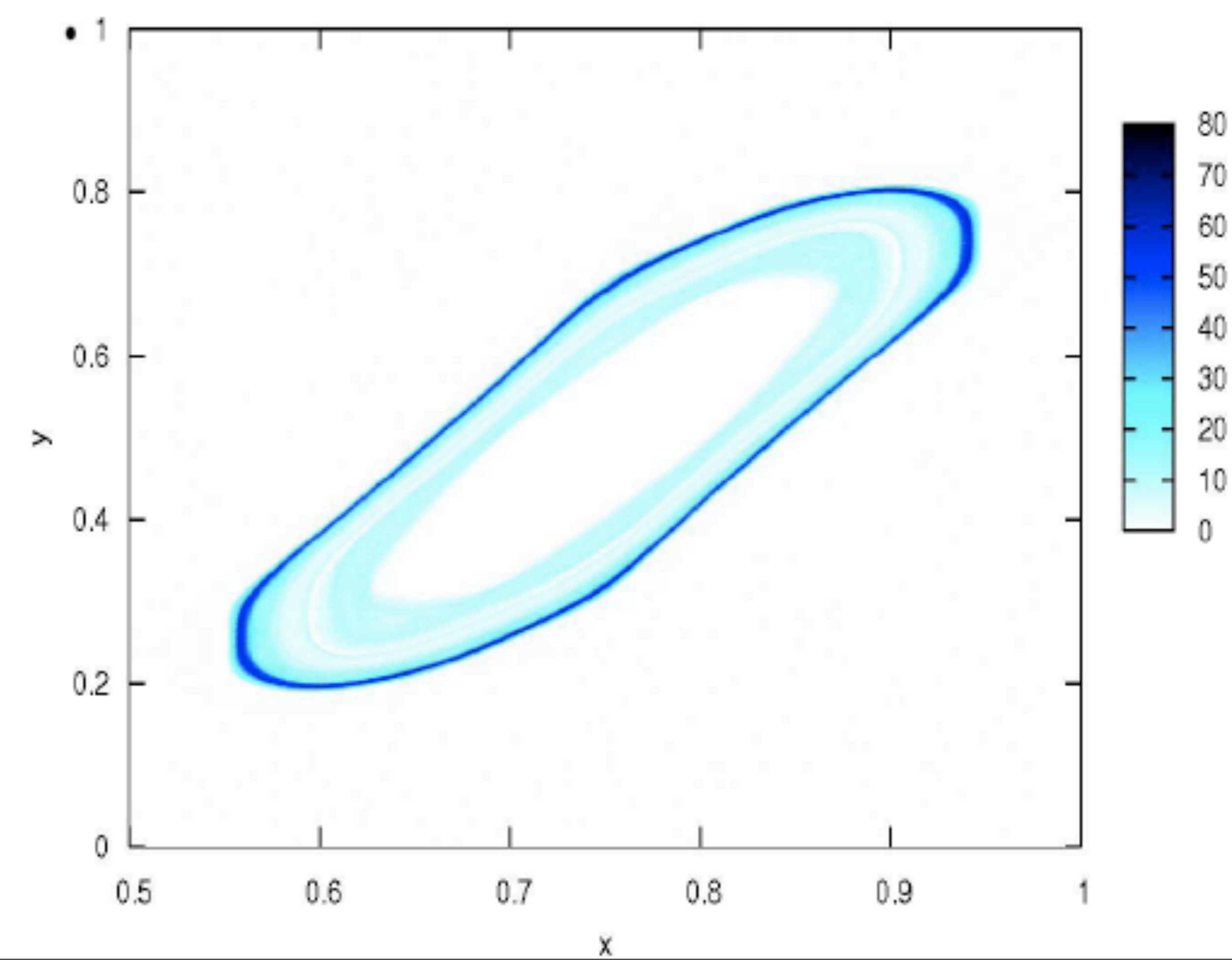


Coupled symplectic maps model



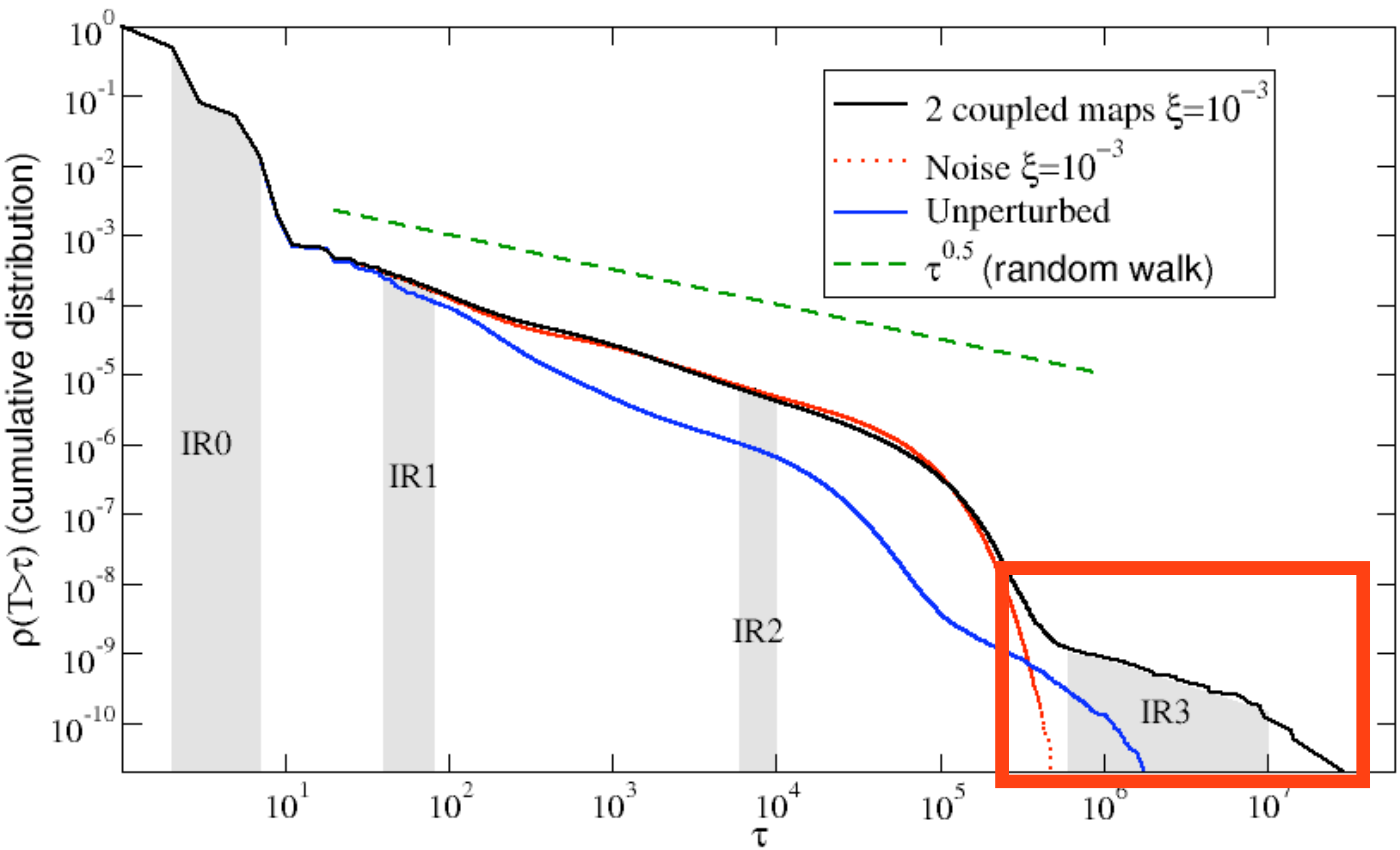






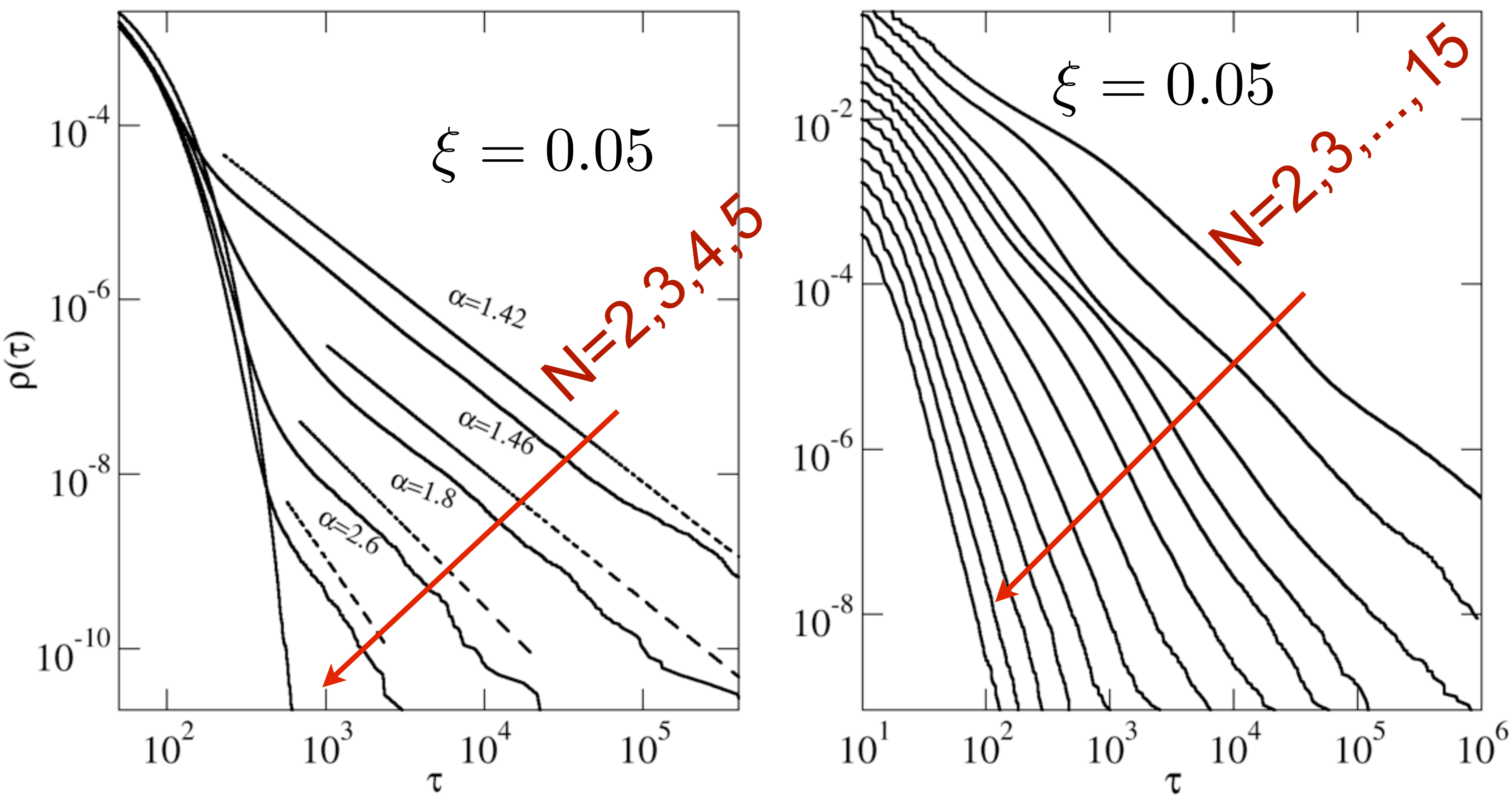
Coupled symplectic maps model

Strong mixing?



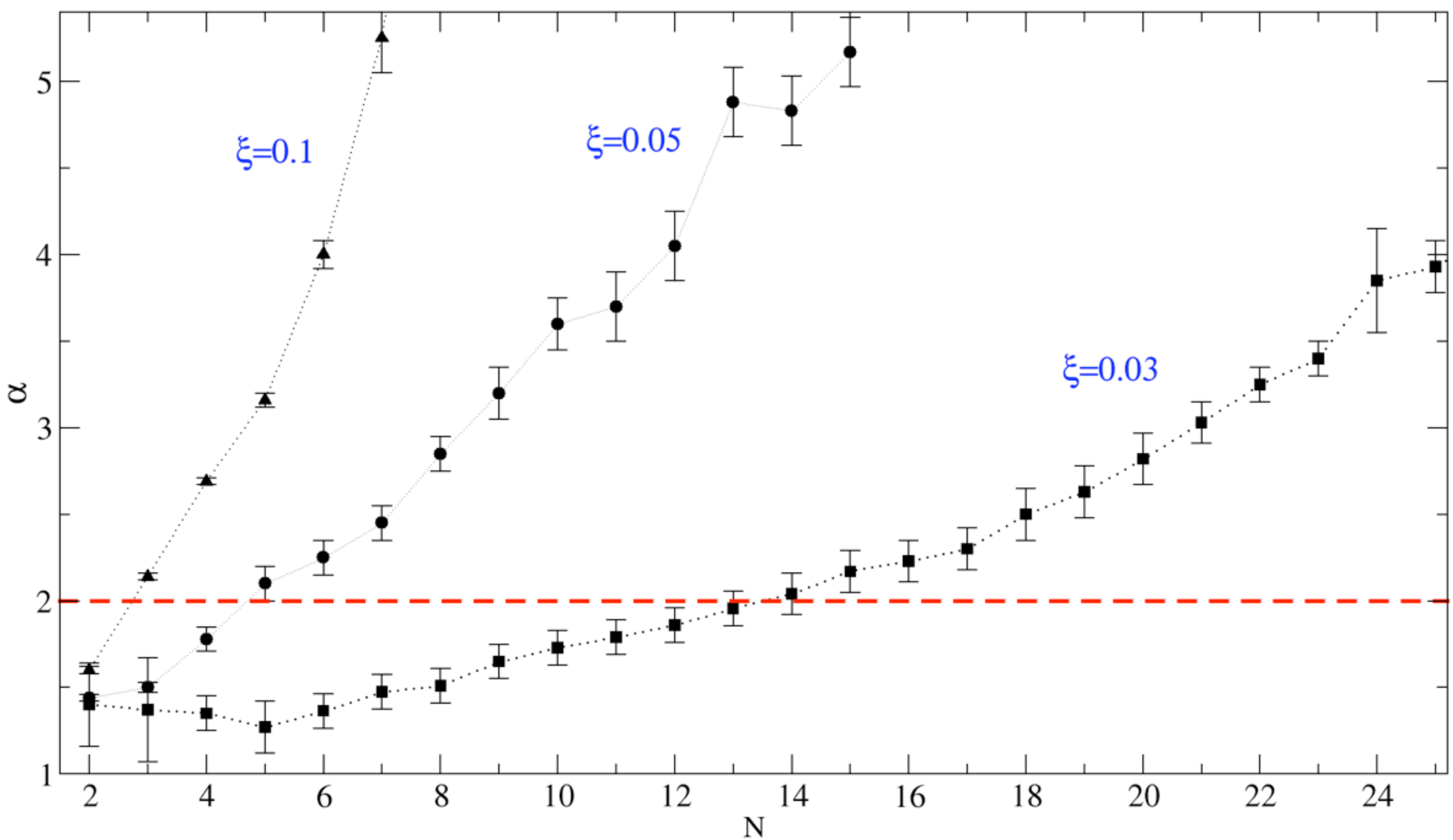
Coupled symplectic maps model

Strong mixing?



Coupled symplectic maps model

Strong mixing?



Coupled symplectic maps

1. Ergodicity, i.e., negligible measure of regular components ✔

2. Strong mixing, i.e., fast decay of correlations ✔

Non-exponential decay, but sufficiently fast power-law

Apresentação V:

fluído incompressível

Passive scalar field $\theta(\vec{x}, t)$ (contaminant), advected by a flow with velocity field given by $\vec{v}(x, t)$ [Aref,1984]

$$\frac{\partial \theta}{\partial t} + \nabla.(\vec{v}\theta) = D_m \nabla^2 \theta,$$

where D_m is the molecular diffusion coefficient. The motion of fluid elements (Lagrangian description) is written as

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}, t) + \eta(t),$$

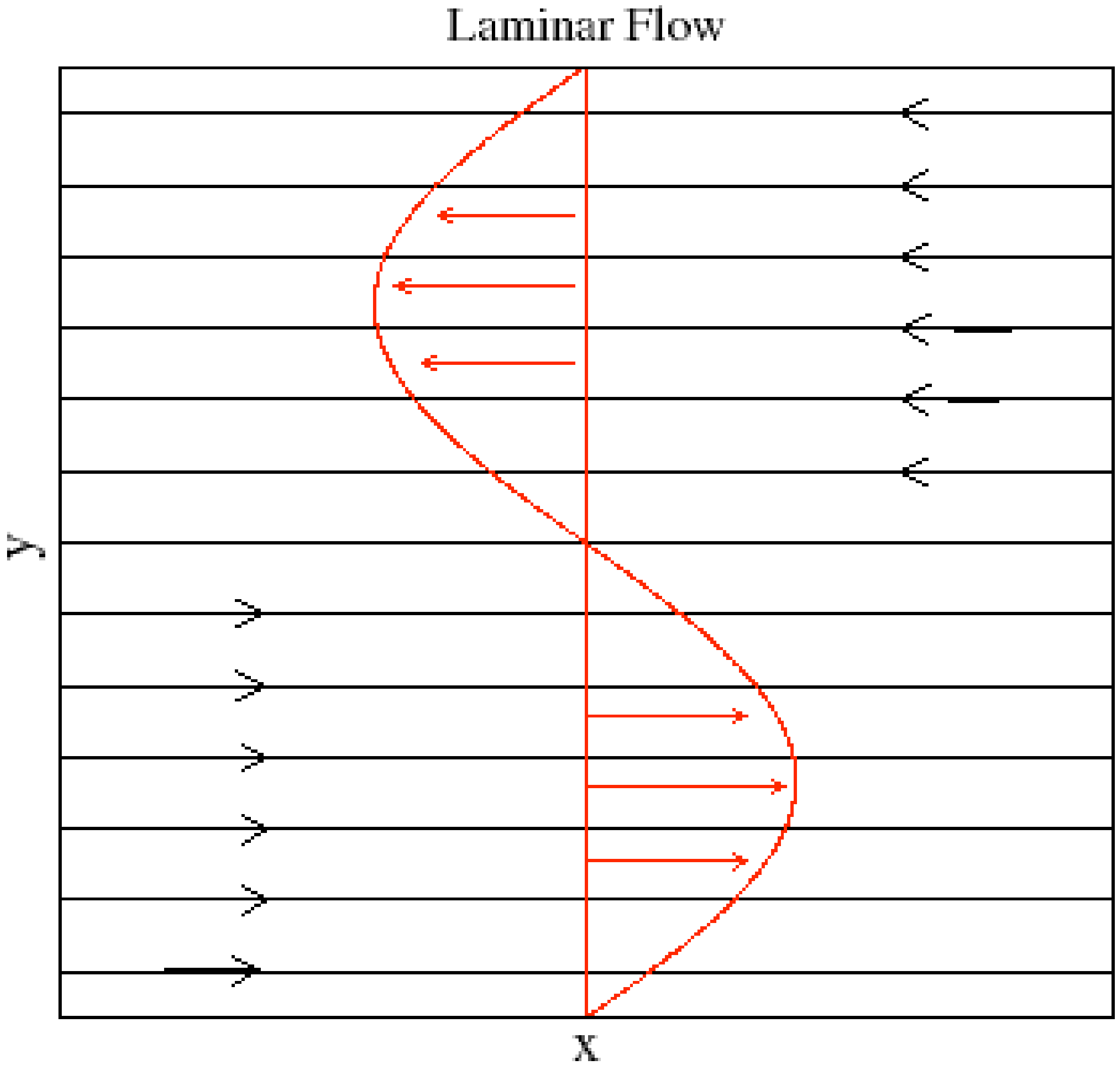
where $\langle \eta_i(t)\eta_j(t')\rangle = 2D_m \delta_{i,j}\delta(t - t')$.

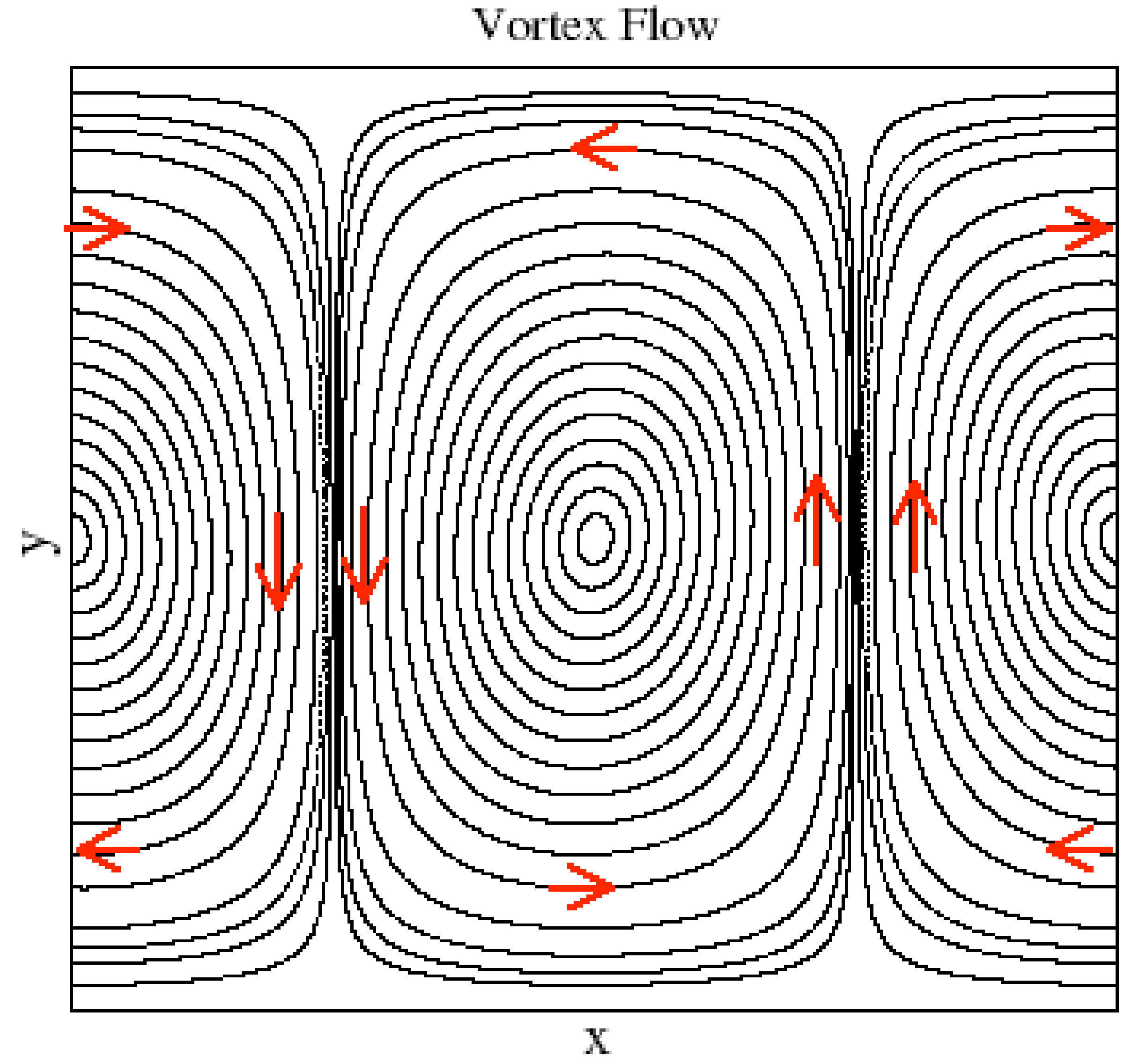
Consider an incompressible $\nabla.\vec{v} = 0$ 2-D fluid $\vec{x} = (x, y)$.

In this case there exist a stream function $\psi(x, y, t)$ such that

$$\frac{dx}{dt} = v_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{dy}{dt} = v_y = \frac{\partial \psi}{\partial x}.$$

Consider a fluid channel infinite in the x direction having the following two flows:
Laminar regime: $\psi_1(x, y) = -v_1 \sin(\pi y)$; Vortex regime: $\psi_2(x, y) = v_2 cos(2x)(1 - y^2)^2$





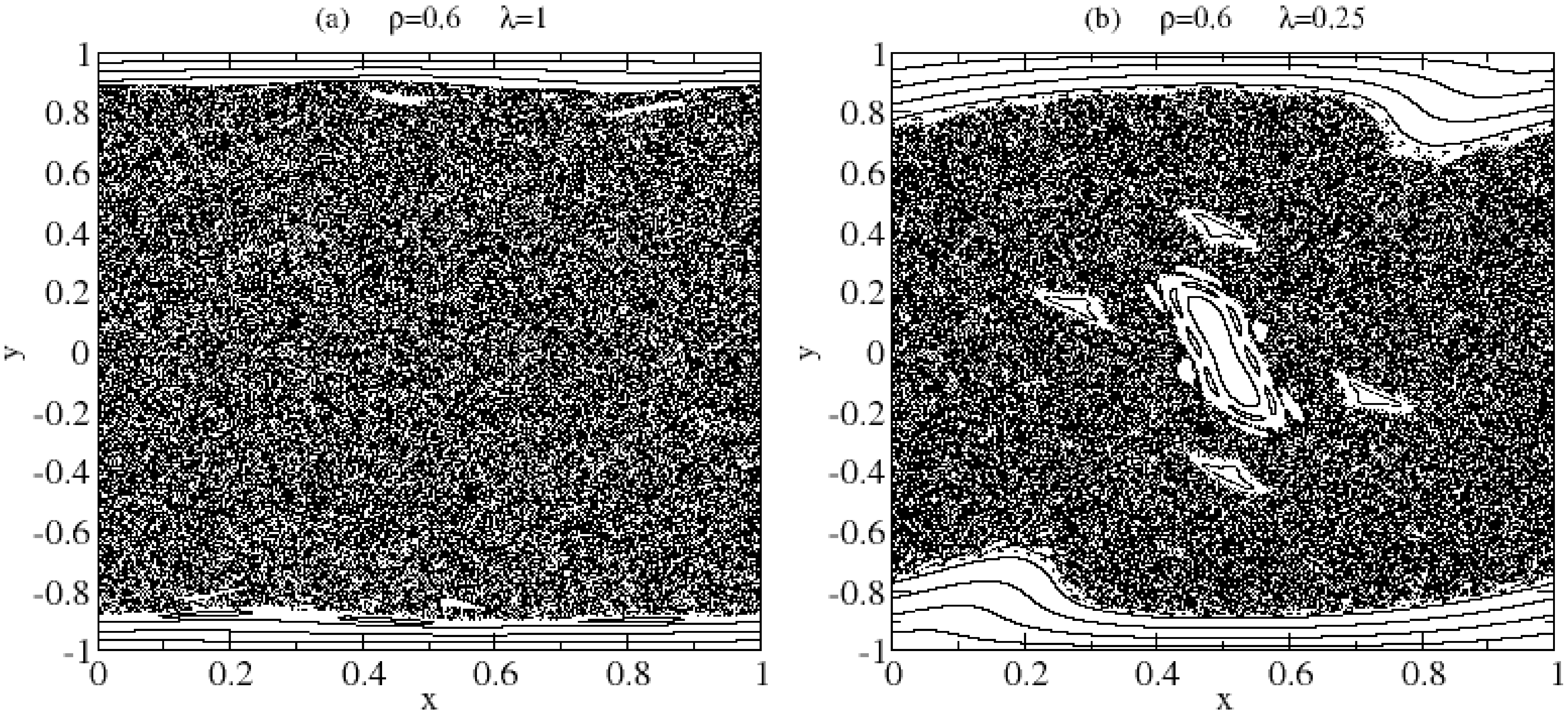
Alternating periodically between the two regimes in a period t_0 and mapping the evolution from $nt_0 \to (n+1)t_0$ one gets

$$
\begin{aligned}
x_n &= x_{n+1} + \lambda \sin(\pi y_n) - \tfrac{2\rho}{\pi} y_n (1 - y_n^2) \cos[2\pi(x_n + 1)] + \xi \delta_n, \\
y_{n+1} &= y_n - \rho(1 - y_n^2)^2 \sin[2\pi x_{n+1}] + \xi \delta'_n.
\end{aligned}
$$

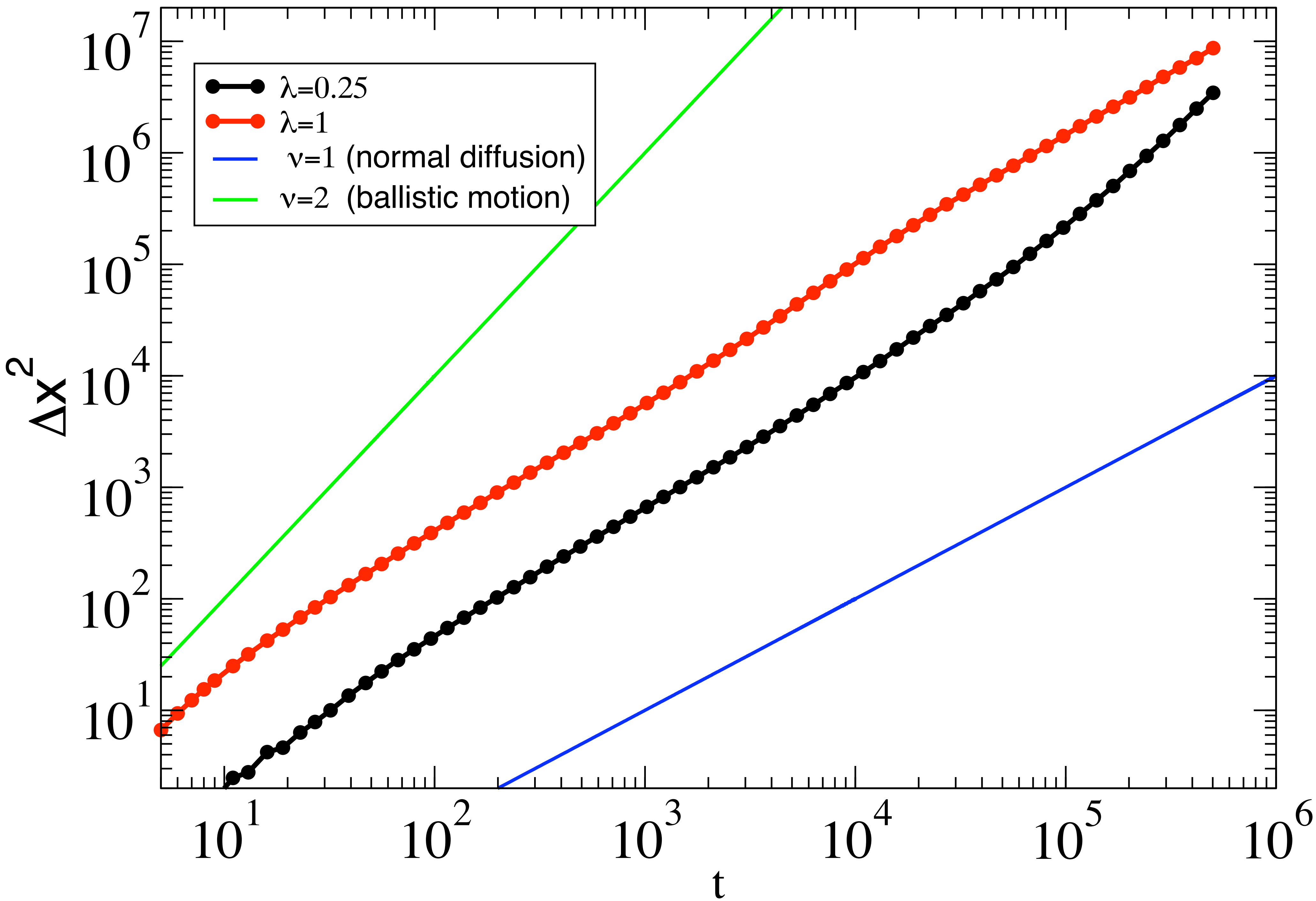
$\rho = \pi v_2 t_0/2$ – intensity of the vortex regime;
$\lambda = v_1 t_0/2$ – intensity of the laminar regime;
ξ – intensity of the white noise variable δ ($\xi \sim \sqrt{D_m}$);

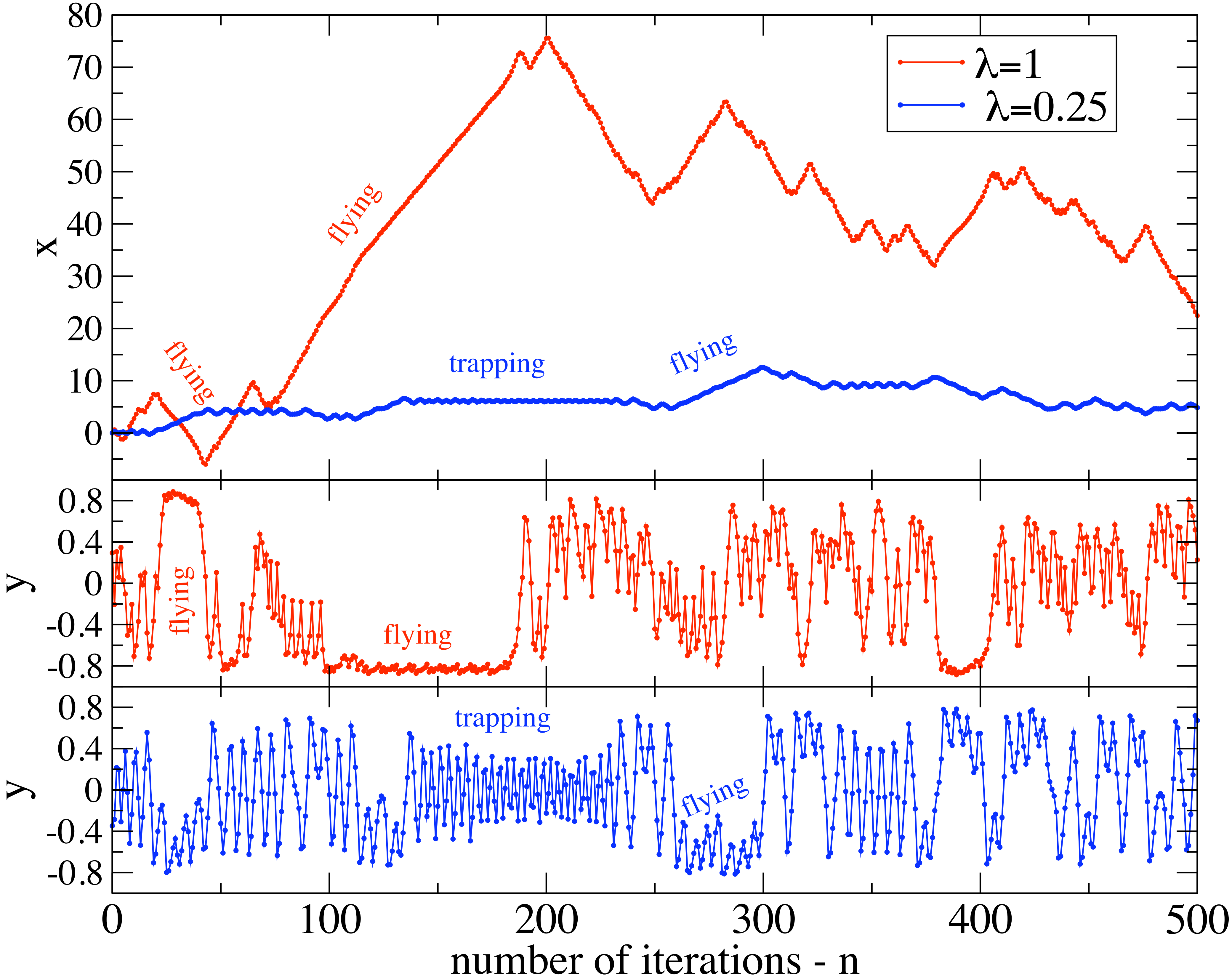
Espaço misto para dois parâmetros de controle

$$
\begin{aligned}
x_n &= x_{n+1} + \lambda \sin(\pi y_n) - \tfrac{2\rho}{\pi} y_n (1 - y_n^2) \cos[2\pi(x_n + 1)] + \xi \delta_n, \\
y_{n+1} &= y_n - \rho(1 - y_n^2)^2 \sin[2\pi x_{n+1}] + \xi \delta'_n.
\end{aligned}
$$



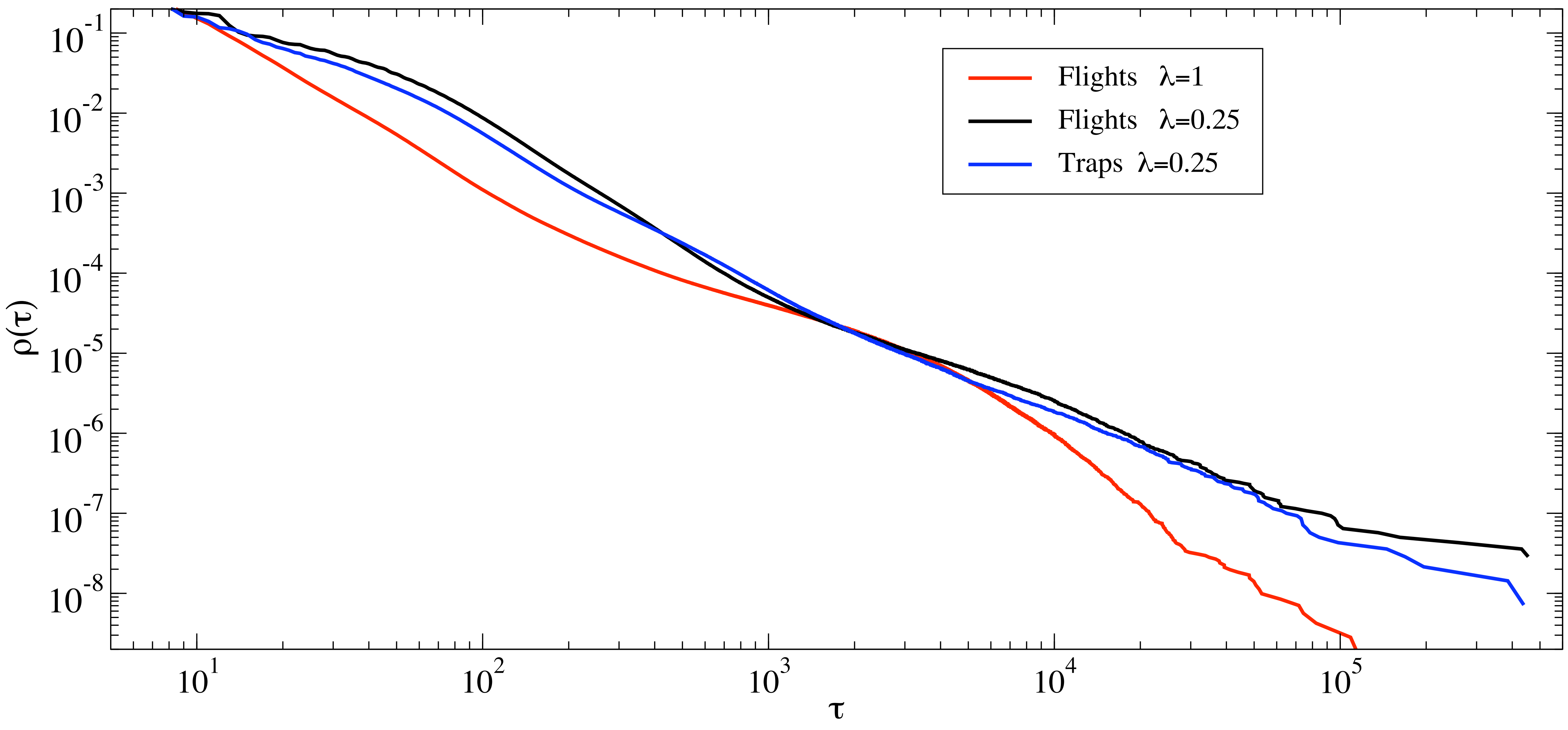
Transporte super-difusivo



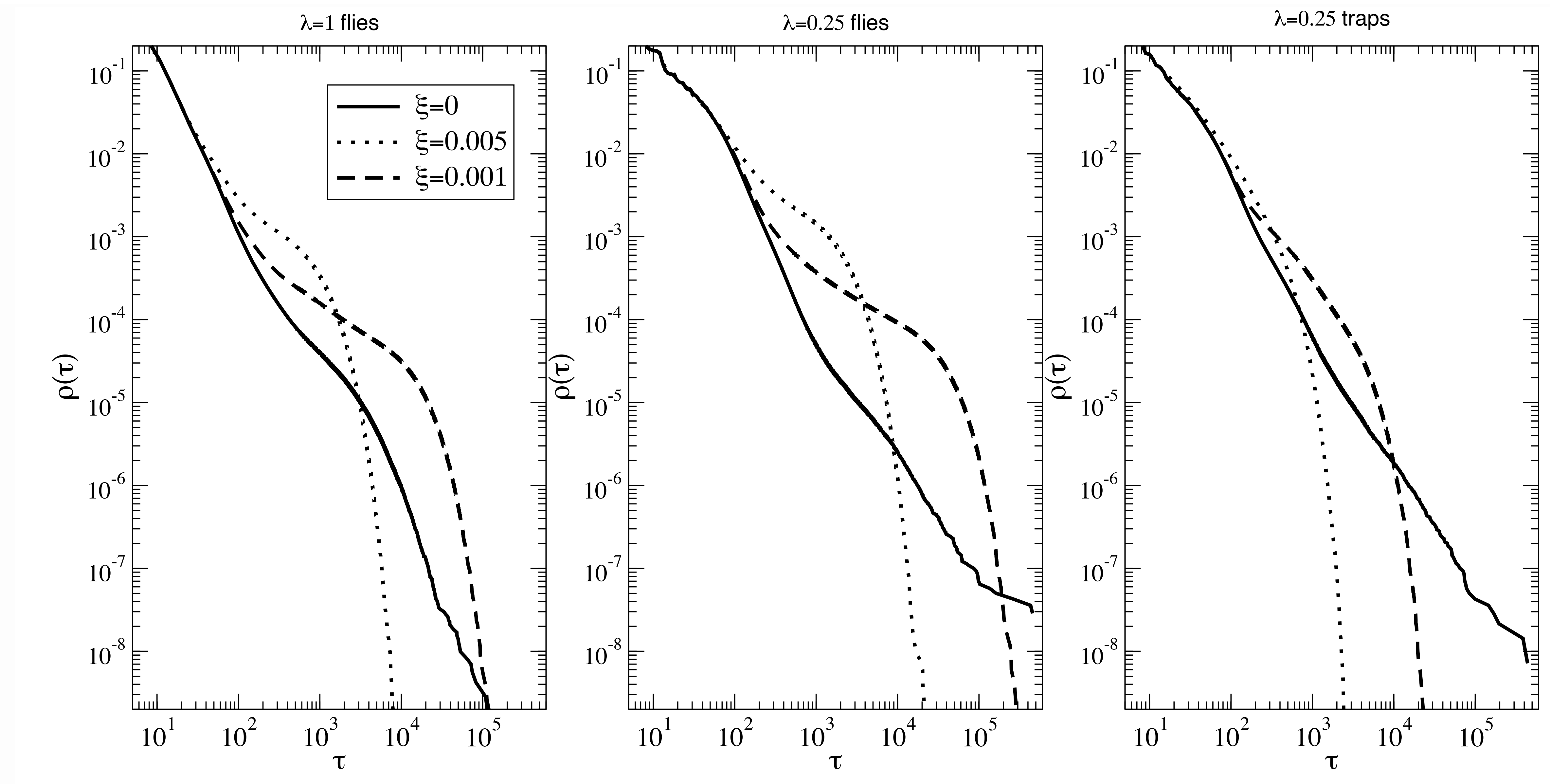


λ=1
λ=0.25
x
y
y
flying
flying
trapping
flying
flying
flying
trapping
flying
80
70
60
50
40
30
20
10
0
0.8
0.4
0
-0.4
-0.8
0
100
200
300
400
500
number of iterations - n

Estatísitca de aprisionamento e voo

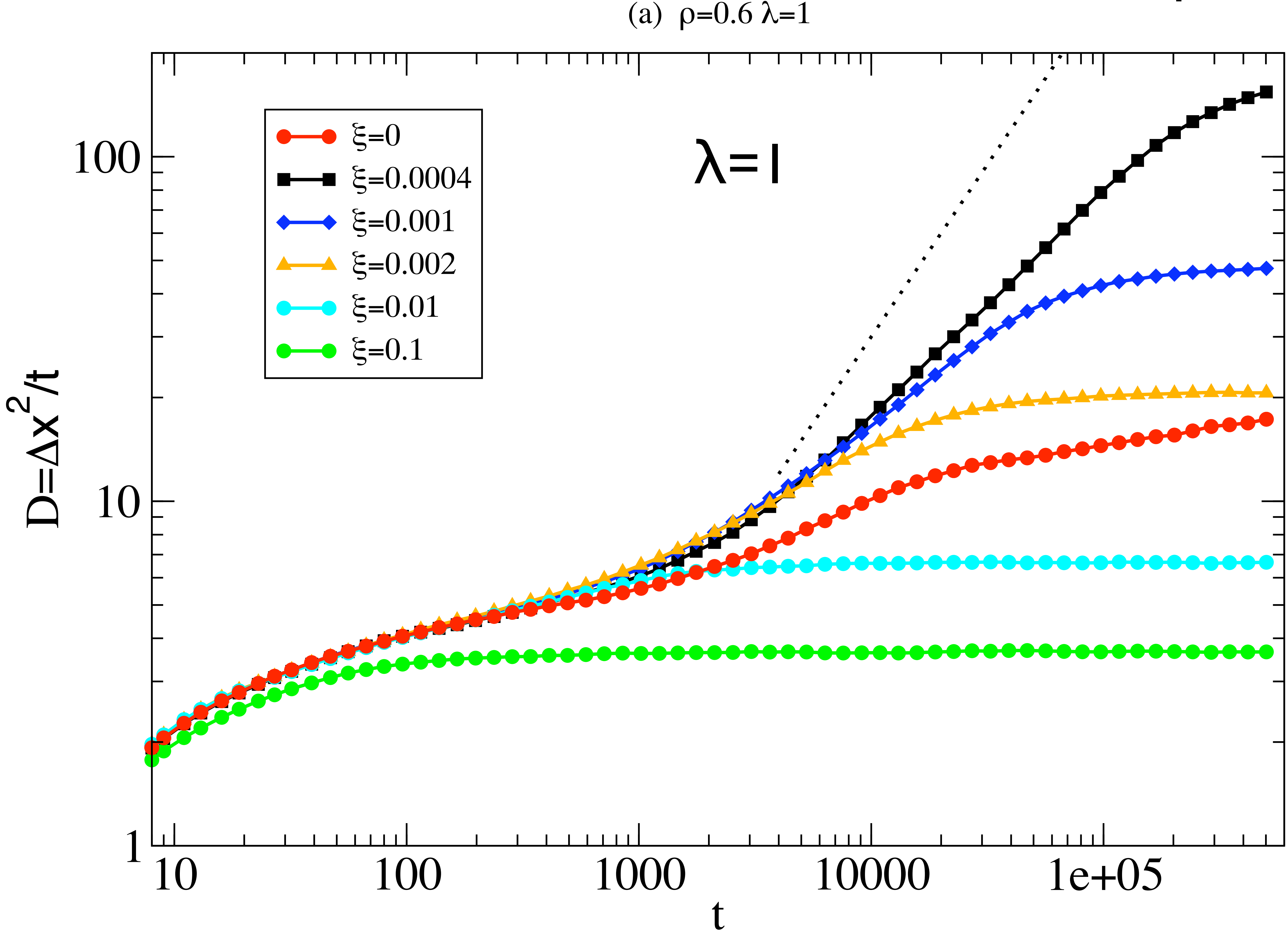


Efeito da difusão molecular no aprisionamento

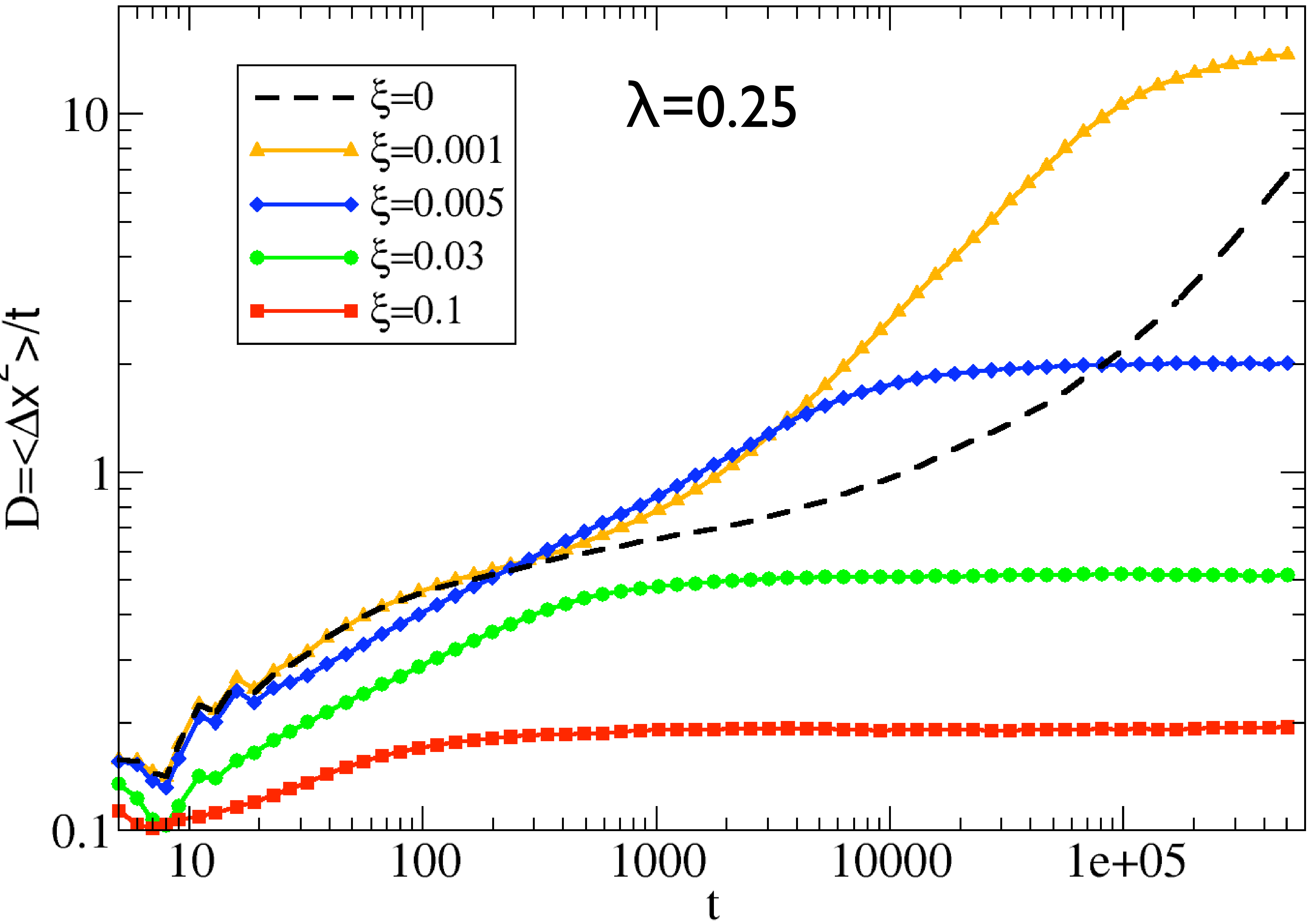


Tempo final do regime de super-aprisionamento $t \sim 1/\xi^2$

Coeficiente de difusão como função do tempo



Coeficiente de difusão como função do tempo



Difusão total (advecção+molecular) como função da difusão molecular